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Patent design

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Prosjektet har mottatt midler fra det alminnelige prisreguleringsfondet.

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Abstract

In this paper we study optimal patent design with sequential innovation. Firms innovate by undertaking "research" activities to generate new ideas and by undertaking "development" activities to transform these ideas into viable products. We characterize the optimal patent policy, with particular focus on the tradeoff between patent length and breadth in this setting.

Innovation is the engine of economic growth. Innovation activities are only undertaken if an innovator can appropriate future benefits from an invention to cover the initial investment cost. In recognition of the possibility that without legal protection there will be insufficient innovation, the patent system was created to promote innovation by granting inventors an exclusive right (patent) to their inventions for a limited period of time. Even though patents foster innovation, facilitate commercialization and improve disclosure, resultant patent monopolies are associated with a variety of distortions. The trade-off between standard deadweight loss and incentives for innovation was the focus of early patent design studies initiated by Nordhaus (1969).[1] They demonstrated how patent length should be chosen carefully in order to balance incentives with deadweight loss.

The two-dimensional analysis used in these early studies, though insightful, is inadequate because both incentives and welfare costs are multidimensional. First, patents are granted *after* invention but *before* commercialization. By centering on pre-invention incentives, the analysis ignores the impact of the patent on incentives for post-invention development. In addition, it only considers one innovator, so welfare costs due to rent-seeking activities such as "patent races", which are often more significant than standard deadweight loss, are not taken into account.[2] Second, there are policy instruments other than patent length, such as patent breadth and patent scope, which are arguably more important and subject to more discretion

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[1]See also Scherer (1972) and Tandon (1982).

[2]Posner (1975) forcefully argues that the producer surplus or rent captured by the monopolist should also be counted as a social cost of monopoly.

than patent length.[3] Third, innovation is often sequential, in the sense that new technologies build on existing ones. In a setting of sequential innovation, there is an additional welfare cost associated with patent monopoly because it may block future improvements. Since the early analysis was conducted in the context of one-time innovation, this additional welfare cost was not taken into account.

Subsequent contributions to the patent design literature address incentives for post-invention development *in isolation*, the trade-off between patent length and patent breadth, and the nature of sequential innovation.[4] This approach is not entirely satisfactory because these issues are closely connected. The goal of this paper is to conduct an integrated analysis of optimal patent policy, providing insight on how different incentives and welfare costs interact and how the level and decomposition of the social value of innovation factor into patent design.

In order to study a patent policy that provides innovators with incentives for both pre-invention research investment and post-invention commercialization investment, and which also takes into account the multidimensional welfare costs associated with patent monopoly, we introduce a model of dynamic innovation whose main components are:

1. Firms innovate by undertaking costly "research" activities to generate new ideas and by undertaking costly "development" activities to turn these ideas into viable products. Competition in research is modeled as "patent races" in order to capture the welfare costs associated with competitive activities in seeking a patent monopoly. Both pre-invention and post-invention incentives are considered in our model.

2. Innovation is sequential and new products supersede old ones. That is, industries experience ongoing series of acts of "creative destruction". Hence, our model considers the additional welfare costs associated with the potential blocking of future improvements on the existing technology, in particular, the trade-off between rewarding current innovators and rewarding future innovators.

3. The social planner chooses a set of instruments, consisting of an expiration time (patent length) and a possibly time-dependent patentability requirement (patent breadth), to maximize the total discounted social welfare, subject to the constraint that she must deliver patent rewards connected with earlier patents.

In our model, a large number of firms undertake "research" activities to generate new ideas which are heterogeneous in quality. We interpret ideas as initial inventions, so the heterogeneity of idea quality captures the fact that inventions may differ in terms of "novelty", "utility", and "non-obviousness". We assume free entry of research firms. A research firm with a new idea can apply for a patent. A patent allows the patentee to prevent its rivals

[3] See Merges and Nelson (1990) for an extensive discussion on the importance of patent breadth and patent scope in patent policy.

[4] We discuss the relevant literature shortly.

from producing its patented product during the period that the patent is valid. Upon patent approval, the patentee then undertakes "development" activities to commercialize the patent.

Patent policy is multi-dimensional. The patent specifies the statutory patent length, it provides protection against imitation (lagging breadth) as well as against improvements that infringe on the patent (leading breadth). Furthermore, for the innovation to be patentable, it must be considered to be a sufficient technological contribution (patentability requirement). As our focus is on sequential innovations, protection against future innovations is a core part of the analysis. The patentability requirement and leading breadth are both instruments that can be used to avoid the erosion of returns on the innovation by subsequent innovations. To that end, leading breadth specifies the set of qualities of the next innovation which infringes on the current patent. A patentability requirement defines the minimum innovation size required to receive a patent.[5] However, there is a significant difference in the way that the patentability requirement and leading breadth impact market dynamics. If a new innovation is patentable but infringes on the current patent, which is the case when the patentability requirement is weak and leading breadth is sufficiently broad, the new patentee cannot commercialize on the innovation without a licence. This situation creates a number of challenges. First, there is a risk that the current patent owner will block or impede subsequent improvements. This yields a potentially severe holdup problem which may negatively impact the innovation rate. Second, if a patent infringes on the previous generation, rent sharing through licensing may strengthen overall innovation incentives as the inventor obtains a share of the profits associated with future innovations, but it may also be detrimental to social welfare due to the risk of collusive behavior. If, however, patentablity requirement is strong and leading breadth is weak, the rent associated with the current patent vanishes as soon as a new innovation occurs. As this paper addresses the impact of patent design on R&D incentives, we assume that the leading patent breadth is narrow. Hence, if a new innovation is patentable, it does not infringe on the current patent.[6]

Lagging breadth protects the innovation against immitation. Conseptually, complete lagging breadth can be considered as a full protection of the innovation's unique contribution, e.g. the return of the entire quality increase beyond the best alternative technology. A narrow protection would allow competitors to enter with a partial imitation, thus with a quality level higher than the previous incumbent producer.[7] In a separate section we discuss the implications of relaxed lagging breadth for innovation incentives. To that end we assume that a relaxed patent breadth materializes as a constraint on the incumbent's price, in that the limit price required to avoid the loss of market share declines.[8]

[5] See O'Donoghue (1998) for a more detailed discussion on the various components of a patent scheme. See also Hunt (2004), Denicolò (2008), Gallini and Scotchmer (2002) and Sena (2004).

[6] This corresponds to the assumptions made in several contributions to the literature, see e.g. Hunt (2004), O'Donoghue (1998) and Hopenhayn et.al. (2006).

[7] See Parello and Spinesi (2005) for a discussion.

[8] See Gilbert and Shapiro (1990), O'Donoghue and Scotchmer (1998), Parello and Spinesi (2005) and Sorek (2011).

Our paper extends and complements earlier analyses of patent scope. Gilbert and Shapiro (1990) and Klemperer (1990) first introduce patent breadth into Naudhaus's framework to investigate the optimal mix of patent length and patent breadth that delivers a given patent reward. In a one-time innovation model with vertical variety, Gilbert and Shapiro show that optimal patents have infinite patent length and different reward sizes can be delivered by varying the patent breadth. In a one-time innovation model with horizontal variety, Klemperer provides conditions under which optimal patent length is finite, and conditions under which optimal patent length is infinite.[9] Both papers, however, acknowledge that the assumption of a single innovation is an important limitation to their analysis, and call for an extension of their analysis to multiple innovations. This paper introduces sequential innovations into the framework of Gilbert and Shapiro (1990), and shows that optimal patents can be finite.[10]

Our analysis also contributes to the rapidly growing literature on sequential innovations. Important papers in this area include Scotchmer (1991, 1996), Scotchmer and Green (1990), Green and Scotchmer (1996), Chang (1995), O'Donoghue (1998), O'Donoghue, Scotchmer, and Thisse (1998), Hunt (2004), Hopenhayn, Llobet and Mitchell (2006) and Bessen and Maskin (2009), among others. O'Donoghue (1998), O'Donoghue et.al. (1998) and Hopenhayn et.al. (2006) are closest to our model. In a sequential innovation setting similar to ours, O'Donoghue (1998) shows that a minimal patentability requirement (patent breadth) can stimulate R&D because it increases the length of market incumbency. However he only considers stationary policies with infinite patent length. We extend his analysis by considering a broader class of patent policies with time-dependent patent breadth and endogenous patent length. We characterize the optimal patentability requirement and the trade-off between patent length and breadth.[11]

O'Donoghue et.al. (1998) find that a broad patent with a short statutory patent life improves the diffusion of new products, while a narrow patent with a long statutory patent life can lower R&D costs. Assuming that idea quality is private information, Hopenhayn et.al. (2006) adopt the mechanism design approach to characterize optimal patent policy and find that the optimal patent policy consists of a menu of patents with infinite lengths but with different breadths.[12] As optimal policy is monotone in type in their model, truth telling is directly implemented. In our model, optimal policy is non-monotone. In a separate section we derive the optimal implementable policy, which requires monotonicity in policy

[9] Gallini (1992) introduces the possibility of costly imitation and shows that broad patents with finite patent lives can be optimal in the one-time innovation setting.

[10] The concept of patent breadth analyzed in Gilbert and Shapiro (1990) and Klemperer (1990) is more of a "lagging breadth" rather than a "leading breadth".

[11] We find that if the patent length is exogenously fixed, the optimal patent breadth is often time-dependent. It is stationary, however, when patent length is endogenously chosen. Hence, the stationarity assumption in O'Donoghue (1998) is not without loss of generality.

[12] The mechanism design approach was introduced to the patent design literature by Cornelli and Schankerman (1999) and Scotchmer (1999).

functions.

Our model builds on O'Donoghue et.al. (1998) and Hopenhayn et.al. (2006), but differs from theirs in at least two significant ways. First, in their papers (see also O'Donoghue (1998)), demand is perfectly inelastic and thus there is no deadweight loss associated with a patent monopoly. In contrast, we allow for a general form of monopoly inefficiency. Thus, the classical trade-off between incentives for innovation and deadweight loss, which is important in our analysis, is absent in their models. Second, both O'Donoghue et.al. (1998) and Hopenhayn et.al. (2006) focus on incentives for post-invention development and ignore incentives for initial invention. In particular, they assume that ideas are free and arrive exogenously. In our model, ideas are costly and the arrival process is endogenously determined by firms' research investments.[13] As a result, our model can capture the additional welfare cost of a patent monopoly in the form of a monopoly rent (or producer surplus) which is dissipated through rivals investments in research. Each feature has important policy implications. For example, unlike in Hopenhayn et.al. (2006), optimal patent length in our model can be finite or infinite.[14]

The paper is organized as follows. The first section of chapter one introduces the building blocks of the model; importantly, we distinguish between pre-invention research and post-invention development. The second section describes the innovation process, which is based on the classic "quality ladder" formulation. In contrast to the classic model, innovation size is considered endogenous, depending on idea quality and patentees' development investments. Chapter two derives the Bellman equation of the problem and formulates the control problem. Chapter three provides the solution to the control problem, and explores the incentive and welfare implications of patent policy. Chapter four shows that with fixed patent length, optimal policy is non-stationary. Chapter five derives the optimal implementable policy assuming that idea quality is private information to the firm. Chapter six concludes.

1 Research and Development

Every innovation starts with a new idea, but new ideas are costly to generate. A large number of homogeneous firms (or new entrants) invest in research to generate new ideas. Ideas are drawn independently from an interval $[0, \bar{z}]$ according to a continuous distribution $\Phi(z)$ with density $\phi(z)$.

We assume that the arrival rate of new ideas for each firm depends on both individual

[13] Banal-Estanol and Macho-Stadler (2010) and Scotchmer (2011) also make a distinction between research and development, but their focus differ significanlty from ours. They investigate how the government should subsidize research and development.

[14] A related paper is Acemoglu and Cao (2010). They study a general equilibrium model with innovation by both existing firms and entrants. The research and development in our model resembles their entrant and incumbent innovation, but we focus on patent design while they focus on the effect of innovation on growth and firm size distribution. Our paper is also related to the analysis of state-contingent patent policy in Acemoglu and Akcigit (2011) and Hopenhayn and Mitchell (2011).

and aggregate research investment, the latter due to a pecuniary externality, explained as follows. Denote by λ_{it} firm i's Poisson flow rate of new ideas at time t. We let λ_{it} be an increasing function of the firm's research investment at time t. More specifically we assume that a research investment of size $b(\lambda_t)\lambda_{it}$, where $\lambda_t = \sum_i \lambda_{it}$ is the aggregate flow rate of ideas, transforms into an individual Poisson rate λ_{it}. The pecuniary externality is represented by the term $b(\lambda_t)$, which is increasing in λ_t, implicitly reflecting that competition for scarce research resources drives up research costs at the margin.

A single firm's investment is assumed to be small compared to the aggregate level, so all firms take λ_t as given. As a result, firm i's idea arrival rate λ_{it} is linear in its own research investment. Research firms are free to enter and exit at any point in time, and there are no fixed entry costs.

Once a firm has an idea, it has to decide whether to develop the idea into a viable product. An idea is either developed immediately or lost, i.e., banking ideas is not possible.[15] Moreover, a product can be freely imitated unless it is patented. Therefore, once a firm generates a new idea, if it is patentable, it will apply for patent and commercialize it.

Commercializing a product is costly. Immediately after the firm has procured a patent it decides how much to invest in development. We assume that development investments and idea quality are complements, in that the marginal cost of development is strictly decreasing with idea quality.

Sometimes we refer to the current patent holder at time t as the incumbent at time t. Similar to O'Donoghue et.al. (1998) and Hopenhayn et.al. (2006) we assume that the incumbent, possibly due to the replacement effect, does not invest in research to generate new ideas to replace its own technology.[16] This assumption helps rule out the possibility of a single firm holding two consecutive patents. This is a rather weak assumption in our setting because, given the large number of research firms, the probability that a particular firm will obtain the next patentable idea is negligible. If patent policy is anonymous (i.e., independent of a firm's identity), the assumption is innocuous; the incumbent has no incentive to invest in research because of the replacement effect and the fact that the expected profit for each research firm is zero. This assumption is made implicitly in most of the patent design literature.

The level of research and development investments, and idea arrivals are private information of the respective firms, but idea quality is assumed to be publicly observable as soon as a firm applies for patent. As shown in Hopenhayn et.al. (2006) patent policy can be implemented with buyout schemes when idea quality is privately observed. But this requires that the policy satisfy a certain monotonicity property. In a separate section we derive the optimal patent policy under this restriction.[17]

[15] See Erkal and Scotchmer (2009) for an analysis which allows innovators to bank ideas for future use.

[16] A similar assumption is made in Klette and Kortum (2004).

[17] Kremer (1998) proposes a clever auction design to determine the private value of patents to facilitate government to buy-outs of patents.

1.1 Cumulative innovations

We adopt a generalized version of the standard multiplicative "quality ladder" formulation pioneered by Aghion and Howitt (1992). In the standard version of this model, quality is represented by a productivity parameter A_n

$$A_n = \theta^n A_0,$$

where n refers to the number of innovations, θ is a fixed parameter $\theta > 1$ and A_0 is the initial value given by history. Hence each time an innovation occurs, productivity increases by factor θ.

In our generalized version each innovation step θ is endogenous, $\theta_\ell = \theta(K_\ell, z_\ell)$, depending on both the patent holder's development investments, denoted K_ℓ, and the innovation specific idea quality z_ℓ. This yields the dynamics

$$A_n = \left[\prod_{\ell=1}^{n} \theta_\ell\right] A_0.$$

For notational simplicity we normalize A_0 to 1.

We assume that the innovation step function $\theta(K_\ell, z_\ell)$ is continuous and increasing in both arguments. Furthermore, we assume that idea quality and developments investments are complements. Hence,

$$\frac{\partial\theta(K_\ell, z_\ell)}{\partial K_\ell} \geq 0, \quad \frac{\partial\theta(K_\ell, z_\ell)}{\partial z_\ell} \geq 0, \quad \frac{\partial^2\theta(K_\ell, z_\ell)}{\partial z_\ell \partial K_\ell} \geq 0. \tag{1}$$

We also assume that an idea quality of zero is worthless and that development investments are essential:

$$\theta(K_\ell, 0) = \theta(0, z_\ell) \leq 1. \tag{2}$$

The model proceeds as follows. Let $q(P)$ be a downward sloping demand curve, and denote by $c(P)$ the associated (Marshallian) consumer surplus defined by

$$c(P) = \int_P^\infty q(\zeta)\, d\zeta.$$

Let mc be a constant marginal cost. We normalize social surplus at marginal cost, denoted by s, to one:

$$s = c(mc) = \int_{mc}^\infty q(\zeta)\, d\zeta = 1. \tag{3}$$

Let $\pi(P) = (P - mc)\, q(P)$ denote producer surplus, and $\chi(P)$ the triangular deadweight loss. With social surplus in first best normalized to one, $c(P)$ and $\pi(P)$ have the interpretations as the respective *fractions* of social surplus that accrue to consumers and firms, given the consumer price P. If price exceeds marginal cost, a fraction $\chi(P) \equiv 1 - c(P) - \pi(P)$ is lost.

All flow terms associated with the n'th innovation are scaled up by the productivity parameter A_n. Hence, with marginal cost pricing, social surplus (and hence consumer surplus) associated with the n'th innovation is

$$s_n = A_n \int_{mc}^{\infty} q(\zeta) \, d\zeta = A_n \equiv \left[\prod_{\ell=1}^{n} \theta_\ell \right]. \tag{4}$$

If price P_n exceeds marginal cost the consumer surplus is

$$c_n = A_n \int_{P_n}^{\infty} q(\zeta) \, d\zeta \equiv \left[\prod_{\ell=1}^{n} \theta_\ell \right] c(P_n) = c(P_n) \, s_n. \tag{5}$$

As is standard, we assume that research costs and development costs follow the same multiplicative process as value added. This formulation is convenient as firms' optimization problems, across innovations, are identical up to a multiplicative constant – both in their roles as research firms and as patentees commercializing ideas. Hence we use n to refer to the $n'th$ patent in the sequence, but whenever we solve an optimization problem we remove the n index for notational simplicity.

If patent n is not active (has expired), the product price drops to marginal cost, and thus $c_n = s_n$. If patent n is active, the flow profit π_n for the patent holder follows from the solution of the incumbent firm's monopoly pricing problem, subject to the constraint set by the consumers' option to buy the previous product generation. Assuming Bertrand competition across product generations, consumers' outside option can be represented by the value of consuming the old product at marginal cost, that is s_{n-1}, ref (4). Since $s_n = \theta_n s_{n-1}$ and $c_n(P) = c(P) \, s_n$, the constraint the new incumbent faces, $c_n(P) \geq s_{n-1}$, can be written

$$\theta(K_n, z_n) \int_{P}^{\infty} q(\zeta) \, d\zeta \geq 1. \tag{6}$$

The formulation of the constraint (6) conforms with complete lagging breadth, consumers' best alternative is represented by the value of consuming the old product. Hence, except for the deadweight loss of monpoly pricing, the entire flow value of the quality improvement is extracted by the innovator. In section 3.3 we relax this assumption by replacing consumers' outside option with a utility flow $u(z) \geq 1$. This impacts the new incumbent's pricing strategy, as the limit price declines. Lagging breadth is complete if $u(z) = 1$.

We can now solve the model by backward induction, starting with the patentee's price optimization problem in the market. Then we go backwards and solve the patentee's development investment problem. Finally we characterize equilibrium research with the free entry of research firms aspiring to be the next incumbent.

For a given development investment K and idea quality z, the patent owner solves the following price setting problem in the market:

$$\max_P (P - mc) q(P) \quad s.t. \;\; \theta(K, z) \int_{P}^{\infty} q(\zeta) \, d\zeta \geq 1. \tag{7}$$

If the constraint binds, P is a " limit price" implicitly determined from the constraint, thus strictly increasing in innovation size $\theta(K,z)$

$$P^L = P\left(\theta(K,z)\right).$$

If the constraint does not bind, the optimal price is the monopoly price P^M, which is independent of $\theta(K,z)$. We use π as a generic notation, and refer to π^M and $\pi^L(P)$ whenever appropriate.

1.2 Commercialization incentives

To provide firms with incentives to innovate, the social planner may grant a monopoly right (patent) to a new idea held by a firm, which prevents other firms from producing the patented product for a certain period of time. A patentee has market power over the patented product until its patent terminates. This may happen either because the patent expires, in which case the technology becomes freely accessible and the price immediately drops to the level of marginal cost, or because a non-infringing new patent takes over the market.

Consider first the incentives for the patent holder to invest in development, K. Let ρ^* denote the expected discounted length of the incumbency period for the current patent in equilibrium, which we also refer to as the *patent duration*[18]. ρ^* is endogenously determined as it depends on the idea arrival rate and the details of the patent policy. In section 1.3 we discuss in detail how ρ^* is determined.

As all flow terms are scaled up by the multiplicative productivity parameter A_{n-1}, the patent holder's maximization problem can be represented as follows[19]

$$\max_K \ \rho^*\theta(K,z)\pi\left(P\left(\theta(K,z)\right)\right) - K, \tag{8}$$

the profit flow $\theta\pi$ over horizon ρ^*, the expected monopoly period, net of investment costs K.

Denote by $K(\rho^*, z)$ the optimal solution. To simplify notation we refer to equilibrium innovation size as $\theta(\rho^*, z) = \theta(K(\rho^*, z), z)$. Note that the extent to which a firm is exposed to competition from previous product generations impacts development incentives, as the limit price $P^L = P\left(\theta(K,z)\right)$ is increasing in K. From (8) it follows directly that K is a strictly increasing function of ρ^*.

As a reference case we let demand function be iso-elastic and we assume that θ takes the following Cobb-Douglas specification,

$$\theta(K,z) = K^\beta z,$$

[18] O'Donoghue et.al. (1998) refer to the expiration time as the *statutory patent life* and the patent duration as the *effective patent life*.

[19] As the firm's net return from development investments is

$$\begin{aligned} &\left[\rho\theta(K,z)\pi\left(P\left(\theta(K,z)\right)\right) - K\right] A_{n-1} \\ &= \left[\rho\theta(K,z)\pi\left(P\left(\theta(K,z)\right)\right) - K\right] \prod_{\ell=1}^{n-1} \theta_\ell \end{aligned}$$

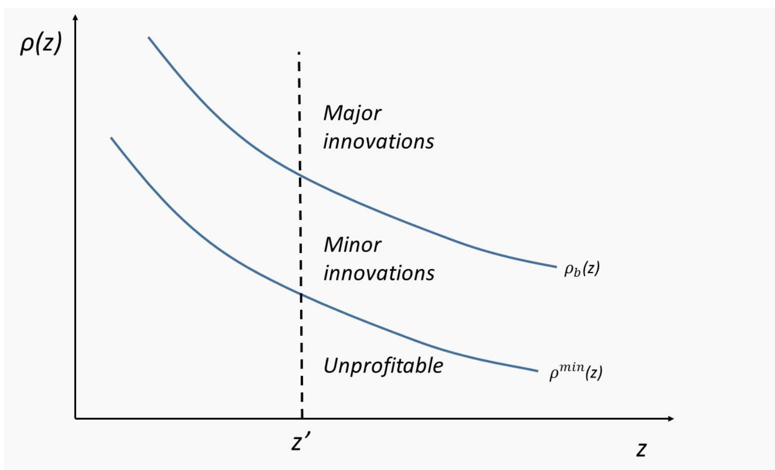
the maximization problem does not depend on the specific history.

where β is a fixed parameter, $\beta \in (0,1)$. Solving (8) yields the following equilibrium levels of development investments in, respectively, the unconstrained monopoly case and in the case of limit pricing,

$$K^M(\rho^*, z) = (\rho^* z \beta \pi_M)^{\frac{1}{1-\beta}}$$
$$K^L(\rho^*, z) = \left(\frac{\rho^* \beta}{z^{\frac{1}{\varepsilon-1}}}\right)^{\frac{\varepsilon-1}{\varepsilon-1+\beta}}.$$

$\varepsilon > 1$ refers to the absolute value of the demand elasticity[20]. Note that K^L is declining in z, reflecting a particular rent extraction effect associated with limit pricing. Furthermore, patent duration has a stronger marginal impact on investments in the monopoly case than if the firm charges a limit price[21]. Both features are discussed in section 3.3.

Figure 1 illustrates the combinations of idea quality z and patent duration $\rho(z)$ that give rise to monopoly price and limit price respectively. From now on we refer to drastic innovations as *major innovations,* and to innovations that yield limit pricing as *minor innovations.*



Consider idea level z'. Since K is increasing in ρ^*, if patent duration is sufficiently long, a firm in equilibrium will charge the monopoly price. Let $\rho_b(z)$ denote the level of patent duration at which $\theta(\rho_b(z), z)c(P^M) = 1$. Thus in the figure, the curve $\rho_b(z)$ delineates major and minor innovations.

If patent duration is too short, it will not be profitable to commercialize the idea. Let $\rho^{\min}(z)$ be the shortest duration associated with idea quality z compatible with a non-negative profit[22], thus:

$$\max \ \left[\rho^{\min}(z)\theta(K,z)\pi\left(P\left(\theta(K,z)\right)\right) - K\right] = 0.$$

From the properties of the θ-function, it follows that $\rho^{\min}(z)$ and $\rho_b(z)$ are both downward sloping, as depicted in the figure.

[20] See appendix for the calculations.

[21] Note that $\frac{1}{1-\beta} > \frac{\varepsilon-1}{\varepsilon-1+\beta}$

[22] As θ is concave in K, and development costs are linear, drastic innovations are always profitable, hence $\rho^{\min}(z) < \rho_b(z)$.

1.3 Patent policy and research investments

Patent policy consists of patent length T and a future sequence of cut offs representing the patentability requirements.

It is clear from (8) that patent duration ρ^* plays a central role in a patentee's investment decision. Recall that patent duration is the *expected* discounted duration of patent monopoly *in equilibrium.* For a fixed z, duration ρ^* uniquely determines equilibrium net profit for the patentee. In other words, conditional on ρ^* and z, a patentee's net profit does not depend on the details of future patent policy, but only on the expected duration of the monopoly period.

In solving the optimal patent design problem, it is therefore convenient to define patent duration as policy instrument. Thus we think about durations as *rewards*, and in doing so we have to take into account that a given reward imposes a constraint on subsequent policy. More precisely, the continuation game induced by future patent policy must be consistent with the specified reward. From now on we refer to $\rho_t(z)$ as the reward of idea quality z arriving at time t[23]. This means that patent policy is described by patent length T and a sequence of patent reward functions $\{\rho_t\left(z\right)\}_{t=0}^{\infty}$ for future ideas, one reward function for each time period, where the reward is the discounted length of the period that the patent remains valid.

Since $\rho^{\min}(z)$ is the smallest reward compatible with non-negative profit, we define the planner's choice set of rewards as follows: $\rho(z) \in \left\{0, \left[\rho^{\min}(z), \frac{1}{r}\right]\right\}$[24], reflecting the binary choice between providing a sufficient reward or letting the idea go, where the latter option is referred to as $\rho(z) = 0$. For a reward to be sufficient, $\rho(z)$ must be selected from the interval $\left[\rho^{\min}(z), \frac{1}{r}\right]$. Patentability requirement can be represented by the treshold rule, in which all ideas z above treshold $\hat{z}$ are patentable, i.e. $\rho(z) \geq \rho^{\min}(z)$ for all $z \geq \hat{z}$, and $\rho(z) = 0$ for all $z < \hat{z}$.

Let t from now on denote the time since current patent z_n was approved. Thus, each time a patent is filed, the time index is reset to zero.

The patent monopoly may end either because the patent expires or a new and better product is discovered and patented, and replaces the old product in the market. The arrival rate of new ideas will depend on research firms' investments, which in turn, will depend on the patent duration that the planner assign to future ideas.

Denote by u_t the arrival rate of patentable ideas at time t. Given patent reward function $\rho_t\left(\cdot\right)$, the arrival rate of *patentable* ideas at time t is the arrival rate of ideas, λ_t, multiplied with the probability that an idea is patentable, $1- \Phi\left(\hat{z}_t\right)$,

$$u_t\left(\rho_t\left(\cdot\right)\right) \equiv \lambda_t\left(\rho_t\left(\cdot\right)\right)\left(1 - \Phi\left(\hat{z}_t\right)\right). \tag{9}$$

[23] Patent duration plays a similar role in our setting as the "promised agent continuation utility" in the dynamic contracting framework (see for example, Spear and Srivastrava, 1987, Sannikov, 2008).

[24] Note that $1/r$ is the maximal possible reward, because rewarding $1/r$ is equivalent to granting the patent holder a perpetuatal monopoly.

Since reward impacts research incentives, the arrival rate of ideas λ_t will depend on the reward function $\rho_t(\cdot)$. We will describe shortly how λ_t relates to $\rho_t(z)$.

The expected duration of patent n measured in discounted time can be written

$$\rho_n^* = \int_0^{T_n} e^{-\int_0^s (u_\tau + r) d\tau} ds$$

where T_n denotes the (statutory) expiration time of patent n. The promise-keeping constraint can now be formulated as follows

$$\rho_n^* = \int_0^{T_n} e^{-\int_0^s (u_\tau + r) d\tau} ds \geq \rho_n, \tag{10}$$

Thus, patent policy can be identified as a combination of a sequence of rewards of the next innovation $\{\rho_t(\cdot)\}_{t=0}^{\infty}$ and patent length T_n, satisfying (10)[25]:

$$\left\{ (T_n, \{\rho_t(\cdot)\}_{t=0}^{\infty}) : \int_0^{T_n} e^{-\int_0^s (u_\tau + r) d\tau} ds \geq \rho_n \right\}.$$

Note that T_n is the time at which patent n expires, whereas $\rho_t(\cdot)$ is the reward of the next innovation.

The timing of the game between the n-th and $(n+1)$-th patent approval is the following. Immediately after a firm files for patent z_n and obtains protection ρ_n (i.e., $t = 0$), the planner announces a future patent policy that is consistent with ρ_n.

Then the current patent holder chooses the *lump-sum* development investment K_n, and in every future period $t > 0$, research firms decide whether to enter and how much to invest in research. The aggregate *flow* investment in research incurred by all research firms in period t generates new ideas with aggregate arrival rate λ_t. If no patentable idea arrives before the expiration date T, the current patent expires at time T. If a research firm gets a patentable idea z_{n+1} at time t, either before T or after T, the firm files a new patent and obtains protection $\rho_t(z_{n+1})$, and the old patent z_n, if not yet expired, is worthless[26]. The new patentee becomes the new patent holder. Now the new patent protection becomes $\rho_{n+1} = \rho_t(z_{n+1})$. The planner then announces a new policy that is consistent with ρ_{n+1}, and the process repeats.[27]

Next consider a research firm's incentive to invest in research As flow terms are scaled up by $\prod_{\ell=1}^{n-1} \theta_\ell = s_{n-1}$, the instantaneous payoff for firm i at time t is $\lambda_{it}\Pi(\lambda_t)$, where $\Pi(\lambda_t)$ is net revenue per dollar research investment given by

$$\Pi(\lambda_t) = \left\{ \int_{z \geq \hat{z}_t} \left[\begin{array}{c} \rho_t(\xi)\, \theta(\rho_t(\xi), \xi)\, \pi(P(\rho_t(\xi), \xi)) \\ -K(\rho_t(\xi), \xi) \end{array} \right] \phi(\xi)\, d\xi - b(\lambda_t) \right\} s_{n-1}, \tag{11}$$

[25] We will show that the optimal policy has the threshold property, in that there exists a cutoff idea $\hat{z}_t$ such that $\rho_t(z) = 0$ if and only if $z < \hat{z}_t$.

[26] Note that the new patent is noninfringing by assumption.

[27] Alternatively, the planner could commit to a patent policy that specifies a fully history-contingent sequence of expiration time and reward functions for all future patentable ideas, not just the next one. It is equivalent to our current formulation where the planner has to choose a new patent policy after every new patent approval.

and where $b(\lambda_t)$, the unit cost of research, is increasing in aggregate research.

Free entry then implies that

$$\begin{aligned} \Pi(\lambda_t) &\leq 0 \text{ if } \lambda_{it} = 0, \\ \Pi(\lambda_t) &= 0 \text{ if } \lambda_{it} > 0. \end{aligned} \tag{12}$$

Therefore, if the equilibrium aggregate research investment λ_t is positive, we can derive $\lambda_t(\rho_t(\cdot))$ as a functional of the reward function $\rho_t(\cdot)$ at time t. It follows from the free-entry condition (12) that

$$\int_{z \geq \hat{z}_t} \left[\rho_t(\xi)\,\theta(\rho_t(\xi),\xi)\,\pi(P(\rho_t(\xi),\xi)) - K(\rho_t(\xi),\xi)\right]\phi(\xi)\,d\xi = b(\lambda_t), \tag{13}$$

which determines the aggregate idea arrival rate λ_t as function of patent policy, $\lambda_t(\rho_t(\cdot))$.

As argued in Posner (1975), the producer surplus or the rent captured by the monopolist should also be counted as part of the social cost of monopoly. In our model, the aggregate research cost incurred in order to obtain a patent is, in free entry equilibrium, equal to the expected producer surplus generated by the patent, as shown in the above rent-dissipation condition (13). However, different from the monopoly model considered in Posner (1975), the incurred research cost here generates new ideas – a socially valuable by-product.

2 The Patent Design Problem

Suppose the current patent has idea quality z_n and is promised patent duration ρ_n. The social planner is utilitarian and designs patent policy in order to maximize the sum of the discounted payoff to incumbents, entrants (research firms) and consumers, subject to the promise-keeping constraint.

The planner's continuation value is the expected discounted value of a stochastic sequence of future innovations. Each innovation is associated with the life expectancy of its product, beginning at the point in time the idea is patented, and ending when the next patentable idea arrives and the product is replaced by a new product in the market. This time span can be divided in two parts. The first part is the expected incumbency period, in which the patent owner extracts monopoly profit. The second part is the expected duration of the period after the patent expires and before the next innovation, during which consumers enjoy the product at its marginal cost.

Furthermore, due to free entry, all producer rents are dissipated in the long run. Thus, in equilibrium, aggregate research expenses correspond exactly to the future expected discounted profit flows, net of development expenses. Accordingly, with a common discount rate, the planner's continuation value is represented by the discounted flow of future consumer surplus flows associated with the innovation sequence.

The welfare function can thus be written as a sum of three terms (a detailed derivation of this expression is given in the appendix)

$$
\begin{aligned}
W_n = & \int_0^\infty \left\{u_t \mathbb{E}_{z\geq\hat{z}_t}\left[W_{n+1}\left(\rho_t\left(\xi\right),\xi\right)\right]\right\} e^{-\int_0^t (u_\tau + r)d\tau} dt \\
& + \int_0^{T_n} c_n\left(\rho_n, z_n\right) e^{-\int_0^t (u_\tau + r)d\tau} dt \\
& + e^{-\int_0^{T_n}(u_\tau + r)d\tau} \int_{T_n}^\infty s_n\left(\rho_n, z_n\right) e^{-\int_0^t (u_\tau + r)d\tau} dt.
\end{aligned}
\tag{14}
$$

To explain the three terms, with Poisson flow rate u_t a patentable idea ξ arrives at time t, and yields a new innovation, with the associated continuation value $W_{n+1}\left(\rho_t\left(\xi\right),\xi\right)$. This explains the first term. The second term is the expected discounted social value of the incumbency period, under which the social surplus corresponds to consumer surplus due to rent dissipation. The third term has the following interpretation: with the probability $e^{-\int_0^{T_n}(u_\tau + r)d\tau}$ no patentable idea arrives before the current patent expires at T_n. At this time, the price falls to the marginal cost, and maximum social surplus s_n is realized.

It is clear from (14) that the reward constraint (10) will hold with equality, since otherwize the planner can reduce T_n and gain from marginal cost pricing over a longer period. Hence, substituting out from the reward constraint (10), we can write the Bellman equation as follows

$$
\begin{aligned}
& W_n\left(\rho_n, z_n\right) \\
= & \max_{\rho_t(\cdot),T} \int_0^\infty \left\{u_t \mathbb{E}_{z\geq\hat{z}_t}\left[W_{n+1}\left(\rho_t\left(\xi\right),\xi\right)\right] + s_n\left(\rho_n, z_n\right)\right\} e^{-\int_0^t (u_\tau + r)d\tau} dt \\
& - \rho_n\left[s_n\left(\rho_n, z_n\right) - c_n\left(\rho_n, z_n\right)\right].
\end{aligned}
\tag{15}
$$

Note that the term $\rho_n\left[s_n\left(\rho_n, z_n\right) - c_n\left(\rho_n, z_n\right)\right]$ is interpreted as the "Posner deadweight loss", as it corresponds to the sum of producer surplus and the deadweight loss triangular,

$$
\rho_n\left[s_n\left(\rho_n, z_n\right) - c_n\left(\rho_n, z_n\right)\right] = \rho_n\left[\pi_n\left(\rho_n, z_n\right) + \chi_n\left(\rho_n, z_n\right)\right].
$$

The Posner deadweight loss represents a pure fixed cost in the planner's problem.

Denote by ω_n the normalized continuation value, defined by

$$
\omega_n = \frac{W_n}{s_{n-1}},
$$

which inserted in (15) yields,

$$
\omega_n\left(\rho_n, z_n\right) = \theta\left(\rho_n, z_n\right)\left[\Omega(\rho_t\left(\xi\right)) - \rho_n\left(1-c\right)\right] \tag{16}
$$

where $\Omega(\rho_t\left(\xi\right))$ denotes the *gross* continuation value at time t,

$$
\Omega(\rho_t\left(\xi\right)) := \int_0^\infty \left\{u_t \mathbb{E}_{z\geq\hat{z}_t}\left[\omega_{n+1}\left(\rho_t\left(\xi\right),\xi\right)\right] + 1\right\} e^{-\int_0^t (u_\tau + r)d\tau} dt. \tag{17}
$$

Subtracting the reward cost $-\rho_n\left(1-c\right)$ yields the associated *net* continuation value. Note that $c = c(P^M)$ if the current innovation is drastic, and $c = c(P(\rho, z))$ if the incumbent is constrained and charges a limit price.

As the planner's problem is independent of the particular stage n in the sequence of innovations, we remove the n index from now on. We then let ρ refer to the current reward, and let $\rho_t(z)$ refer to the reward of the next innovation, as it may depend on quality z, and arrival time t.

We can then write optimization problem (15) as follows

$$\Omega^*(\rho) = \max_{\rho_t(\cdot),T} \Omega(\rho_t(\xi)) \quad s.t. \int_0^T e^{-\int_0^t (u_\tau + r)d\tau} dt \geq \rho, \tag{18}$$

which is a maximization problem we can solve using standard control theory.

With no further restrictions on policy, the solution of this problem is referred to as the first best policy. We also study the problem by imposing two constraints on policy. First the constraint that T is fixed, which corresponds to standard patent practice. Second we derive the optimal design by imposing a monotonicity constraint on the reward function, corresponding to the implementability condition in Hopenhayn et.al. (2006).

From the solution of (18) the net continuation value can be written

$$\omega(\rho, z) = \theta(\rho, z) [\Omega^*(\rho) - \rho(1-c)]. \tag{19}$$

Note that the only state variable in our problem is the current reward ρ.

Define Ω^{**} as the unconstrained optimum

$$\Omega^{**} = \max_{\rho_t(\cdot),T} \Omega(\rho_t(\cdot)). \tag{20}$$

Since the cost associated with the current reward is a fixed cost in the planner's problem, it follows that Ω^{**} is the gross continuation value in the case with no active patent (in which case there is no reward constraint), as well as in the case where the reward constraint is not binding, thus[28]

$$\Omega(\rho) = \Omega^{**} \text{ if } T < \infty. \tag{21}$$

Hence the planner's optimization problem *with* an active patent *and* a non-binding reward constraint (that is finite T), is formally equivalent to the optimization problem *without* an active patent. As mentioned, this follows directly from the observation that the social cost associated with a non-binding reward constraint is a pure fixed cost. Thus, if in optimum patent length T is always finite, the patent optimization problem would be represented by a sequence of independent "static problems". However, in situations where the optimal patent length is infinite, current patent policy imposes a constraint on future patent policy, giving rise to distortions.

3 Control problem - optimal patent policy

We solve the problem (18) using optimal control theory. Let's define the state variable $x(t)$ as

$$x(t) = e^{-\int_0^t (u_\tau + r)d\tau}$$

[28]This is confirmed by the solution of the control problem, see below.

with boundary conditions $x_0 = 1$ and x_∞ free, and where

$$u_t \equiv \lambda_t \left(\rho_t \left(\cdot\right)\right) \Phi_{\eta t}$$

from (9).

It follows that

$$\dot{x}\left(t\right) = -\left(u_t + r\right) x\left(t\right).$$

Let us introduce another state variable $y\left(t\right)$ which is defined as

$$y\left(t\right) = \int_0^t x\left(s\right) ds.$$

Then we can replace the PK constraint (10) by

$$\dot{y}\left(t\right) = x\left(t\right) \text{ with boundary conditions } y_0 = 0 \text{ and } y_\infty \geq \rho.$$

where the situation with no active patent is interpreted as $\rho = 0$.

As we want to prove that a threshold policy is a feature of an optimal design, we adopt a slightly more general formulation in the control problem. We introduce a binary variable $\eta_t(z)$ which takes the value 1 if idea quality z is provided a patent, and zero otherwise. The probability that an idea arriving at time t is rewarded, and thus implemented, is denoted $\Phi_{\eta t} = \int \eta_t(z)\phi(z)dz$. Finally, we denote by Υ_t the set of ideas for which $\eta_t(z) = 1$.

Let $\rho_t(z)$ and T be the control variables, where $\rho_t(z)$ is a piecewise continuous function of t on $[0, \infty)$. (x_t, y_t) are the state variables. We refer to $\rho_t^*(\cdot), T^*, x_t^*, y_t^*$ as the optimal control. We can write the planner's problem as

$$\begin{aligned}
& \max_{\rho_t(\cdot),T} \int_0^\infty \left\{u_t \mathbb{E}_{\xi\in\Upsilon_t}\left[\omega_{n+1}\left(\rho_t\left(\xi\right),\xi\right)\right] + 1\right\} x_t dt & (22)\\
\text{subject to } : \; & \dot{x}_t = -\left(u_t + r\right) x_t & (23)\\
: \; & \dot{y}_t = x_t & (24)\\
: \; & x_0 = 1 \text{ and } x_\infty \text{ free} & (25)\\
: \; & y_0 = 0 \text{ and } y_\infty \geq \rho & (26)\\
: \; & \rho_t(z) \in \left\{0, \left[\rho^{\min}(z), \frac{1}{r}\right]\right\} \text{ for all } z,\;\; T \in [0,\infty). & (27)
\end{aligned}$$

This is an infinite horizon optimal control problem.

The Hamiltonian is given by

$$\begin{aligned}
& H\left(x, y, \rho_t(\cdot), T, p_1, p_2, t\right) \\
= \; & \left[u_t \mathbb{E}_{\xi\in\Upsilon_t}\left[\omega_{n+1}\left(\rho_t\left(\xi\right),\xi\right)\right] + 1 - p_1\left(t\right)\left(u_t + r\right) + p_2\left(t\right)\right] x_t \qquad (28)
\end{aligned}$$

where $p_1\left(t\right)$ and $p_2\left(t\right)$ are a pair of continuous and piecewise differentiable adjoint functions associated with (23) and (24), respectively.

Define $\widehat{H}(x, y, p_1, p_2, t)$ as the maximum value of the Hamiltonian when $\{\rho_t(\cdot), T\}$ is chosen optimally:

$$\widehat{H}(x, y, p_1, p_2, t) \equiv \max_{\rho_t(\cdot), T} H(x, y, \rho_t(\cdot), T, p_1, p_2, t) . \tag{29}$$

It is trivial from (28) that $\widehat{H}$ is linear in x and y, and thus concave for all t. Therefore, we can apply the infinite horizon version of the Arrow sufficiency theorem.[29]

According to the Arrow sufficiency theorem, $\{\rho_t^*(\cdot), T^*\}$ is optimal if there exist $p_1(t)$ and $p_2(t)$ such that the following conditions are satisfied:

$$\dot{p}_1(t) = -\{u_t \mathbb{E}_{\xi \in \Upsilon_t}[\omega_{n+1}(\rho_t(\xi), \xi)] + 1 - p_1(t)(u_t + r) + p_2(t)\} \tag{30}$$

$$\dot{p}_2(t) = 0 \tag{31}$$

$$[\rho_t^*(\cdot), T^*] \in \max_{\rho_t(\cdot), T^*} H(x^*, y^*, \rho_t(\cdot), T, p_1, p_2, t) \tag{32}$$

$$\lim_{t \to \infty} [p_1(t)(x_t - x_t^*) + p_2(t)(y_t - y_t^*)] \geq 0 \tag{33}$$

where (30) and (31) are adjoint equations, (32) is the maximum principle, and (33) is the transversality condition, which must hold for all admissible x_t and y_t. Note that $\lim_{t \to \infty} x(t) = 0$ for all $x(t)$ since $u_t \geq 0$. Furthermore, $p_2(t) \leq 0$ with $p_2(t) = 0$ if $\lim_{t \to \infty} y_t > \rho$ for all admissible $y(t)$. Hence the transversality condition is always satisfied. It follows that we only need to take care of the adjoint equations and the maximum principle.

Proposition 1 *Given reward ρ, there exists an optimal policy $\rho_t^*(z), T^*$ for all $t \in [0, \infty)$ with the following characteristics: i) The optimal reward function is stationary: $\rho_t^*(z) = \rho^*(z)$, ii) The optimal policy is a threshold-policy, and there exists a unique stationary threshold, $\hat{z}$, such that ideas below $\hat{z}$ are non-pantentable, and ideas above $\hat{z}$ are patentable, iii) $\rho^*(z)$ is non-monotone and discontinuous at $z = \tilde{z}$, where $\tilde{z}$ is the idea level at which the firm charges a limit price for all $z < \tilde{z}$, and the monopoly price for all $z > \tilde{z}$, iv) $\frac{d\hat{z}}{d\rho} > 0$ and $\frac{d\rho^*(z)}{d\rho} \leq 0$ if ρ is binding, $\frac{d\hat{z}}{d\rho} = 0$ and $\frac{d\rho^*(z)}{d\rho} = 0$ otherwise, T^* is infinite if the reward constraint binds, otherwize T^* is finite.*

Proof. See appendix. ■

Stationarity implies that neither the optimal reward of the next innovation nor the threshold depend on whether the current patent has expired or not. Technically this reflects the fact that the (Posner) deadweight cost of monopoly is a fixed cost, $\rho(1-c)$, i.e. the expected patent duration multiplied with the deadweight loss flow. Not surprisingly, a fixed cost does not impact optimal control. Thus the *gross* continuation value is equal both with and without an active patent.

However the endogeneiety of T is essential to this result as policy instruments T and $\rho(z)$ have highly different roles. The rewards of future ideas $\rho(z)$ impact current and future research and development investments, whereas T has no such impact. Hence in optimum,

[29] See Theorem 14 on page 236 of Seierstad and Sydsæter (1987).

$\rho(z)$ is set to maximize the continuation value, whereas T is adjusted such that the promise keeping constraint holds. If T is exogenous, this independence between current and future policy breaks, and as shown in the next section, the optimal policy is non-stationary.

The optimal threshold $\hat{z}$ has the interpretation of the optimal patentability requirement. As time is scarce, rejecting a patent application has an option value. In the control problem the option value is reflected in the two adjoint variables p_1 and p_2. These are closely related, and represent shadow prices associated with the reward constraint of the problem. Both can be given economic interpretations that are useful to our further discussion.

p_2 is the shadow cost of reward, i.e. the impact on the continuation value associated with a marginal change in the reward level,

$$p_2 = \frac{d\Omega^*(\rho)}{d\rho} \leq 0. \tag{34}$$

$p_2 = 0$ if the constraint is not binding, in which case $\Omega^*(\rho) = \Omega^{**}$ from (20).

p_1 has the interpretation of the marginal value of a "unit of discounted time". In optimum p_1 reflects the value of the marginal idea, where the planner is indifferent between providing the idea a patent and letting the idea go (as if it never had existed). Since time is scarce, the latter option has a value, which is reflected in the shadow price p_1.

From the control problem we can derive p_1. Since $\rho^*(\cdot)$ is stationary, and the optimal policy is a threshold policy, from (17) we have,

$$\Omega^*(\rho) = \frac{u\mathbb{E}_{z\geq\hat{z}}\left[\omega\left(\rho\left(z\right),z\right)\right]+1}{u+r}.$$

Therefore, from (30) and from stationarity, which implies that $\dot{p}_1 = 0$, we obtain,

$$p_1(\rho) = \Omega^*(\rho) - \frac{p_2}{u+r}. \tag{35}$$

Since $p_2 = 0$ if the constraint is non-binding, that is if $\rho < \frac{1}{u+r}$, and $p_2 = \frac{d\Omega^*(\rho)}{d\rho}$ otherwise, we can represent (35) by

$$p_1(\rho) = \Omega^*(\rho) - \rho\frac{d\Omega^*(\rho)}{d\rho} \tag{36}$$

(36) defines p_1 as a function of ρ. In the appendix we show that p_1 is increasing in ρ, thus from differentiating (36)

$$\frac{dp_1}{d\rho} = -\rho\frac{d^2\Omega^*(\rho)}{d\rho^2} \geq 0 \tag{37}$$

it follows that $\Omega^*(\rho)$ is declining and concave in ρ. Note that if the reward constraint is not binding, $\frac{d\Omega^*(\rho)}{d\rho} = p_2 = 0$, thus

$$p_1 = \Omega^{**}. \tag{38}$$

Generally, optimal reward and optimal patent breadth are simultaneously determined by the solution of the control problem. However one case is particularly simple, which we therefore will use as a benchmark. If research supply is perfectly inelastic, the idea arrival rate is not impacted by patent policy. Hence the future sequence of idea arrivals is an exogenous process, and the planner's problem is to allocate time optimally across innovations given this exogenous idea generating process. Afterwards, the results are generalized.

3.1 Inelastic research supply

Suppose the research unit cost function $b(\lambda)$, ref the free entry condition (13), has the shape of an " inverse L". That is, the research unit cost is constant up to $\bar{\lambda}$ and then approaches infinity. The arrival rate of ideas will then be constant equal to $\bar{\lambda}$, independent of patent policy.

From the maximum principle, by pointwize maximization, the optimal reward of idea z satisfies the necessary condition

$$\frac{d\omega\left(\rho\left(z\right),z\right)}{d\rho\left(z\right)} \leq 0, \tag{39}$$

where $\rho^{*}(z) = \rho^{\min}(z)$ if (39) holds with strict inequality. Recall that

$$\omega\left(\rho\left(z\right),z\right) = \theta\left(\rho\left(z\right),z\right)\left[\Omega^{*}\left(\rho\left(z\right)\right) - \rho\left(z\right)\left(1-c\right)\right] \tag{40}$$

where

$$c = \min\left[c_M, \frac{1}{\theta\left(\rho\left(z\right),z\right)}\right],$$

depending on whether the idea gives rise to monopoly or limit price. Note that the optimal reward $\rho^{*}\left(z\right)$ of idea z, if patentable, is independent of current reward constraint, as ρ does not impact (40).

We know from Proposition 1 that there is a unique threshold $\hat{z}$. We can therefore characterize the optimal policy as follows: assume first that the reward constraint is not binding, and derive the optimal policy under this assumption. If this candidate policy is compatible with the reward constraint, the policy is certainly optimal. If not, the policy is not feasible, and the reward constraint binds. This yields one more equation, and one additional endogenous variable, p_2.

The candidate policy yields at threshold $\hat{z}$ which satisfies

$$\omega\left(\rho\left(\hat{z}\right),\hat{z}\right) = p_1 = \Omega^{**} \tag{41}$$

since $p_2 = 0$ under the assumption that the reward constraint does not bind.

Recall the interpretation of p_1 as the value of a unit of discounted time. If the government rejects the application (as if the idea had never existed) the continuation value is Ω^{**}, on the right hand side of (41). If the application is accepted, the continuation value is $\omega\left(\rho\left(\hat{z}\right),\hat{z}\right)$, on the left hand side. In optimum the planner is indifferent, thus (41) defines $\hat{z}$ as unique function of Ω^{**}.

If the candidate policy is feasible, meaning that the policy is compatible with the reward constraint, it is the optimal policy. The condition that the policy is compatible with the reward constraint, is that the expected waiting time for the next innovation exceeds reward ρ when $\hat{z}$ is set according to (41),

$$\frac{1}{\bar{\lambda}\left(1-\Phi(\hat{z})\right)+r} \geq \rho. \tag{42}$$

If (42) holds then T is residually determined by:

$$\frac{1}{\bar{\lambda}\left(1-\Phi(\hat{z})\right)+r}\left(1-e^{-\left(\bar{\lambda}(1-\Phi(\hat{z}))+r\right)T}\right)=\rho.$$

Suppose that the candidate policy is not feasible. Then the threshold $\hat{z}$ (which is constant over time since the policy is stationary) is used as a device to screen out a sufficient proportion of ideas to ensure that the promise to the current patentee is kept. Thus $\hat{z}$ is set such that

$$\rho=\frac{1}{\bar{\lambda}\left(1-\Phi(\hat{z})\right)+r}. \tag{43}$$

This is one equation which determines $\hat{z}$ as an increasing function of ρ. Note that for all patentable ideas, the necessary condition for optimal reward is the same as before; $\rho^*(\cdot)$ does not depend on ρ.

Note that the optimal $\hat{z}$ is equivalently derived by the condition

$$\omega\left(\rho\left(\hat{z}\right),\hat{z}\right)=p_1(\rho)=\Omega^*\left(\rho\right)-\rho p_2$$

which corresponds to (41) in the case with a non-binding constraint. Thus the option value associated with rejecting the application is larger when the reward constraint binds since $p_2<0$, reflecting the scarcity of time.

3.2 Elastic research and the research externality

Suppose now that aggregate research supply is elastic, and thus impacted by future rewards. Thus if $\rho(z)$ increases, or the threshold $\hat{z}$ declines, the marginal return from research goes up. This stimulates research and yields a higher aggregate idea arrival rate.

Pointwize maximization yields the following necessary condition for reward of idea quality z

$$\lambda(\rho(z))\frac{d\omega\left(\rho\left(z\right),z\right)}{d\rho\left(z\right)}\phi(z)+\left[\int_{\hat{z}}^{\infty}\left[\omega\left(\rho\left(\xi\right),\xi\right)-p_1\right]\phi(\xi)d\xi\right]\frac{\partial\lambda}{\partial\rho(z)}\leq 0 \tag{44}$$

The second term captures the impact of a higher future reward on idea arrival rate, see the proof of Proposition 1 for details. $\frac{\partial\lambda}{\partial\rho(z)}>0$ since higher rewards stimulate research, see proof of Proposition 1 for an exact derivation. The term $\int_{\hat{z}}^{\infty}\left[\omega\left(\rho\left(\xi\right),\xi\right)-p_1\right]\phi(\xi)d\xi$ is the *research externality*, i.e. the expected social value of an additional idea. The research externality is always positive which we explain below.

Because of free entry there is full rent dissipation on the firms' side. The social value of research is therefore associated with the pure net externality. This is always positive. To see this, suppose the planner sets the threshold according to the same rule as with inelastic research, that is $\omega\left(\rho\left(\hat{z}\right),\hat{z}\right)=p_1$. Suppose that a new idea arrives. If this idea has quality z, its net social value is $\omega\left(\rho\left(z\right),z\right)-p_1$. Since the planner rejects any applications for which $\omega\left(\rho\left(\xi\right),\xi\right)<p_1$, the research externality, which can now be written $\int_0^{\bar{z}}\max\left[0,\omega\left(\rho\left(\xi\right),\xi\right)-p_1\right]\phi(\xi)d\xi$, is certainly positive.

Due to the research externality, the optimal rewards of future ideas for a given threshold $\hat{z}$, is higher with elastic research supply. The reason is that higher rewards stimulate current research, and since the research externality is positive, the planner raises $\rho(z)$ beyond the optimal level in case of exogenous research, for any given threshold.

A similar logic applies for the threshold. The first order condition for $\hat{z}$ is exressed as (see appendix for details)

$$\omega\left(\rho\left(\hat{z}\right),\hat{z}\right)+\left[\int_{\hat{z}}^{\infty}\left[\omega\left(\rho\left(\xi\right),\xi\right)-p_1(\rho)\right]\phi(\xi)d\xi\right]\Delta\lambda(\hat{z})=p_1(\rho) \tag{45}$$

where $\Delta\lambda(\hat{z})$ captures the marginal impact on the idea arrival rate if $\hat{z}$ is included in the set of patentable ideas. Hence, for a given reward function $\rho(z)$, $\hat{z}$ is set below the level at which $\omega\left(\rho\left(z\right),z\right)$ equals p_1.

We can now characterize the optimal policy by applying the same logic as above. First we solve the unconstrained optimization problem, which yields a candidate policy. If the candidate policy is compatible with the reward constraint, the policy is optimal. If the policy is not feasible, there is one more constraint (the reward constraint) and one more endogenous variable, p_2.

In the unconstrained optimum, $\rho(\cdot)$ and $\hat{z}$ are simultanously determined from (43) and (45). Note that $\rho(\cdot)$ is declining and $\hat{z}$ is increasing in ρ. To see this, we know that $p_1(\rho)$ is increasing in current reward ρ - the opportunity cost of a patent is increasing in the reward constraint. A higher p_1 reduces the value of the research externality, and thus yields both a lower reward and a higher threshold.

Suppose now that the candidate policy is not feasible. Then we have a third equation

$$\frac{1}{\lambda\left(\rho(\cdot),\hat{z}\right)\left(1-\Phi(\hat{z})\right)+r}=\rho \tag{46}$$

and a new variable p_2. Note that the additional impact on $\hat{z}$ from (46) reduces the value of the research externality even further.

As emphasized, if the current reward constraint is non-binding, the patent design problem is essentially a static problem, in the sense that the choice of policy does not create future distortions. Recall that $d\Omega^*(\rho)/d\rho=0$ in such cases, and the reward constraint does not impact the continuation value. Hence if the optimal T is always finite, the optimal patent design will be a sequence of solutions of static optimization problems. However, since the research externality is positive and rewards do not create future distortions in such an equilibrium, the planner will then certainly want to reduce the treshold even further. This yields the following result: suppose that the current reward is *not* binding, hence T is finite. Then there must be a positive probability that the next reward will be binding. More specifically, if T_n is finite, the optimal reward of the *threshold idea* is associated with an infinite patent length:

Lemma 1 *If T_n is finite, then $T_{n+1}(\hat{z})=\infty$*

Proof. See appendix. ■

The result is intuitive. As consumers can buy the old product at marginal cost, the net externality of an innovation on consumers can never be negative. If the current T is finite, allowing the innovation does not exert any future distortions. Consequently, the planner always has an incentive to include more ideas in the set of patentable ideas by reducing the threshold. However at some point, z falls so low that the lowest feasible reward $\rho^{\min}(z)$ requires $T = \infty$. Including more ideas at this point will create future distortions. The optimal threshold balances future distortions with the marginal gain of including even more ideas. Thus the smallest patentable idea must be rewarded with $T = \infty$.

An implication of this result is that the optimal patent reward is non-monotone in idea quality z. This result raises the issue of implementability, ref the discussion in Hopenhayn et.al. (2006). They derive an implementable policy scheme based on idea quality as private information, with the thruth-telling constraint that the reward function is monotone. A monotonically declining reward function is incompatible with truth-telling. In a later section we characterize the optimal implementable policy.

3.3 Lagging breadth and the incentive power of reward

In this section we generalize the analysis by relaxing the assumption of complete lagging breadth. Incomplete lagging breadth yields incentives to enter the market with a partial imitation of the currently best product. Hence consumers' outside options are improved, and we represent the outside option by a non-declining function $u(z)$. The entry threat impacts the incumbent's pricing strategy, and thus the incentives to invest in research and development.

The patentee solves the investment problem (8), which in case of incomplete lagging breadth can be expressed as follows:

$$\max_K \rho\theta(K,z)\pi(P) - K \qquad s.t.\ \ \theta(K,z)\int_P^\infty q(\zeta)\,d\zeta \geq u(z), \tag{47}$$

and where $u(z) = 1$ if lagging breadth is complete.

As a point of reference, let us introduce an indicator of the *incentive power* of the patent scheme, $I(\rho, z)$, implicitly defined by

$$\rho\frac{d\theta(K,z)}{dK}I(\rho,z) = 1. \tag{48}$$

We say that a patent scheme has incentive power $I(\rho, z)$ if the solution of (47) corresponds to (48). $I(\rho, z)$ can be written (see proof of Proposition 2)

$$I(\rho,z) = \pi(P) + \left[1 - \left(\frac{P - mc}{P}\right)\varepsilon(P)\right]c(P).$$

The first observation is:

Proposition 2 *$I(\rho, z)$ is declining in z and ρ and increasing in u*

Proof. see appendix. ■

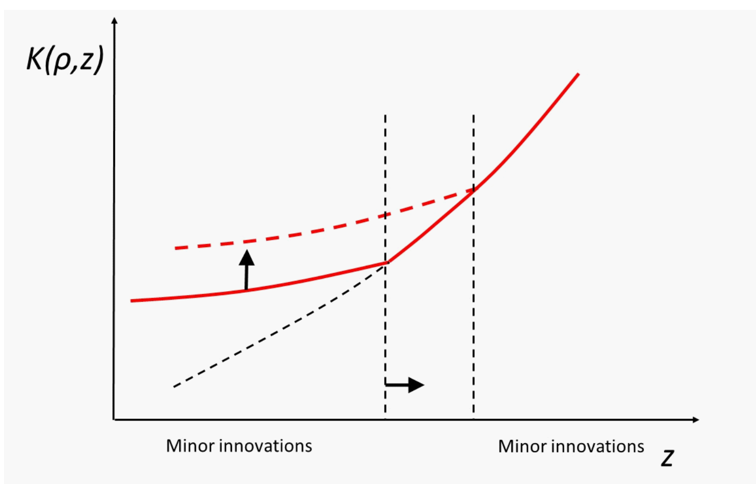
If the firm is unconstrained in its price setting, the first order condition for K is

$$\rho \frac{d\theta(K,z)}{dK}\pi^M = 1 \tag{49}$$

hence $I(\rho, z) = \pi^M$, reflecting the classic (Arrow) appropriability problem, as consumer surplus is not taken into account by the firm. Accordingly, the incentive power is fixed, $I(\rho, z) = \pi^M$, for all combinations of z and ρ supporting the unconstrained monopoly price.

If a firm is constrained in its price setting the picture changes, as limit pricing boosts investment in development. A higher level of development investment allows a firm to raise the price it charges from consumers, since the constraint is relaxed. As the price is below the monopoly level the price increase has a first order effect on profits. Intuitively, the "consumer's participation constraint" binds, as she has the the option to buy the old product at its marginal cost. This rent extraction effect associated with development investment stimulates growth. Hence, all else being equal, a price constrained firm invests more in development than the unconstrained monopoly firm. Thus incomplete lagging breadth induces the incumbent firm to invest more in development.[30]

However, higher incentive power corresponds with a *smaller marginal impact* of reward on investment, illustrated in the following figure:



For a fixed $u(z)$ the solid line represents the firm's optimal development investment[31]. From Proposition 1, we see that the optimal reward function $\rho(z)$ is non-monotone and discontinuous. Discontinuity in the reward function appears at the idea level, which in equilibrium distinguishes between limit pricing and monopoly pricing. Discontinuity can be explained by two features of limit pricing, both associated with rent extraction. Due to the

[30] Obviously, relaxing lagging breadth reduces the expected return from research. Hence there is a trade off between the impact of lagging breadth on pre innovation research and post innovation development.

[31] The figure is based on the technical restriction that $\theta(K, z)/u(z)$ is increasing in z for all K, an assumption that rules out the possibility that consumers' outside option consitutes a binding constraint for a high z and not for a lower z.

kink at $\rho_b(z)$ the optimal reward jumps upwards at the idea level that distinguishes minor from major innovations. The kink is one explanation of this discontinuity. The second reason is related, and can be explained as follows: if the innovation is major, the gain from a higher reward will partly accrue to consumers due to the standard appropriability problem, and partly to the firm. If a firm is constrained in its price setting, and charges a limit price, the firm extracts all gains on the margin. However, this transfer is not neutral. Due to free entry and full rent dissipation, the transfer from consumer surplus to profit represents a welfare loss. Thus, as a higher reward induces a firm to invest more in development, the *direct* welfare effect is positive if the firm is unconstrained, but it is not positive if the firm is constrained. All else being equal, this yields a higher optimal reward for patents associated with major innovations than with minor innovations.

Incentive power $I(\rho, z)$ is declining in ρ and z and incresing in $u(z)$. The reason is that the degree of appropriability is declining. A higher reward stimulates development investment, and under limit pricing the firm extracts consumers' gain by raising the price. However, higher investments also increase the triangular deadweight loss. This effect is negligble if the price is close to the marginal cost, in which case there is almost full appropriability. However, the higher the margin the larger the marginal loss due to an increase in deadweight loss. Thus the degree of appropriability is declining in P. As P is strictly increasing in ρ and z, the result follows.

Relaxed lagging breadth can be represented by a positive shift in $u(z)$. Improved outside option shifts the development investment curve upwards as indicated by the dashed curve. The higher incentive power associated with minor ideas corresponds to the "escape competition effect" discussed in Aghion (2015). Note the different impact that market competition has on *research* investments versus *development* investments in our model. The rent dissipation effect of fierce market competition impacts research negatively. When it comes to development incentives, the impact is quite different, as the rent dissipation effect is transformed to a rent extraction effect which stimulates investments. Thus product market competition and patent protection are complements in relation to post-invention investments. If idea quality z is low, the patentee has a particularly strong incentive to invest in commercialization as it is the only way it can obtain a rent from the patent. Accordingly, from a social point of view, minor ideas have the particular feature that a given investment level can be induced by smaller rewards. However this requires that property rights be strictly enforced such that the return on the innovation is protected.

This challenges the idea that minor ideas should be sorted out by setting a higher patentability requirement, see Hunt (2004) and O'Donoghue (1998). An essential insight in O'Donoghue is that patentability requirements prolong market incumbency, thus increasing innovation rewards; which may spur innovation and growth. However, introducing post-invention commercialization investment to the model introduces a trade off, as the incentive power of rewards depends on innovation size: minor ideas can be developed at a smaller social cost – not only because of higher incentive power, but also because of the lower deadweight loss of monopoly associated with limit pricing.

4 Fixed patent length

Patent systems usually have a fixed statutory expiration time. A fixed T imposes an additional constraint on the optimization problem. We will now solve the control problem with this additional constraint to illustrate the importance of determining patent length endogenously.

The constraint is that the planner, within the given time span T, must adjust the patent instruments such that policy is compatible with the initial reward. In the control problem we therefore replace the terminal condition $y_\infty \geq \rho$ with the condition $y_T \geq \rho$, where T is exogenously given. In order to compare with free T, we refer to the continuation value in the fixed T optimum as Ω_T^*, and to Ω_T^{**} as the unconstrained optimum (corresponding to the continuation value with no active patent). Furthermore, we let $\rho_T^*(z)$ and $\hat{z}_T^*$ denote the optimal unconstrained policy in this regime.

From the state variable

$$x(t) = e^{-\int_0^t (u_\tau + r) d\tau}$$

it follows that

$$x(T) = 1 - \int_0^T [u(t) + r]\, x(t)\, dt.$$

Since the patent expires at time T, if it is not replaced before, we can rewrite the objective function as follows:

$$\begin{aligned} & \int_0^\infty \{u_t \mathbb{E}_{\xi \in \Upsilon_t} [\omega(\rho_t(\xi), \xi)] + 1\} x(t) dt \\ = & \int_0^T \{u_t \mathbb{E}_{\xi \in \Upsilon_t} [\omega(\rho_t(\xi), \xi)] + 1\} x(t) dt + x(T)\Omega_T^{**} \\ = & \int_0^T \{u_t \mathbb{E}_{\xi \in \Upsilon_t} \omega(\rho_t(\xi), \xi) + 1 - [u_t + r]\, \Omega_T^{**}\} x(t) dt + \Omega_T^{**} \end{aligned} \tag{50}$$

where u_t, as before, refers to the patentable idea arrival rate at time t, $u_t = \lambda_t \Phi_{\eta t}$, and where $\Phi_{\eta t}$ is the probability that the idea is patentable. Since Ω_T^{**} is the unconstrained continuation value, the first term of (50) is negative, reflecting the cost side of the active patent. The optimal policy minimizes this cost.

We can now solve the control problem, where we refer to $\rho_t^T(\cdot), x_t^T, y_t^T$ as the optimal control. The planner's problem is

$$\max_{\rho_t(\cdot)} \int_0^T \left\{u_t \mathbb{E}_{\xi\in\Upsilon_t}\omega\left(\rho_t\left(\xi\right),\xi\right)+1-\left[u\left(t\right)+r\right]\Omega_T^{**}\right\} x_t dt$$
$$: \quad \dot{x}_t = -\left[u_t+r\right]x_t$$
$$: \quad \dot{y}_t = x_t$$
$$: \quad x_0 = 1 \text{ and } x_T \text{ free}$$
$$: \quad y_0 = 0 \text{ and } y_T \geq \rho$$
$$: \quad \rho_t(z) \in \left\{0, \left[\rho^{\min}(z), \frac{1}{r}\right]\right\}.$$

This yields a control problem in *finite* time.

We can write the Hamiltonian as follows

$$\begin{aligned} & H\left(x,y,\rho(\cdot),p_1,p_2,t\right) \\ = & \left[u_t \mathbb{E}_{\xi\in\Upsilon_t}\omega\left(\rho_t\left(\xi\right),\xi\right)+1-\left(u_t+r\right)\left(p_1\left(t\right)+\Omega_T^{**}\right)\right]x_t+p_2\left(t\right)x_t. \end{aligned} \tag{51}$$

Define $\widehat{H}\left(x,y,p_1,p_2,t\right)$ as the maximum value of the Hamiltonian when $\rho_t\left(\cdot\right)$ is chosen optimally:

$$\widehat{H}\left(x,y,p_1,p_2,t\right) \equiv \max_{\rho_t(\cdot)} H\left(x,y,\rho_t(\cdot),p_1,p_2,t\right). \tag{52}$$

It is clear that $\widehat{H}$ is linear in x and y, and thus concave for all t. Therefore, we can apply the finite horizon version of the Arrow sufficiency theorem.[32]

According to the Arrow sufficiency theorem, $\rho_t^*\left(\cdot\right)$ is optimal if there exist continuous functions $p_1^T\left(t\right)$ and $p_2^T\left(t\right)$ such that the following conditions are satisfied:

$$\dot{p}_1^T\left(t\right) = -\left\{u_t \mathbb{E}_{\xi\in\Upsilon_t}\omega\left(\rho_t\left(\xi\right),\xi\right)+1-\left(u_t+r\right)\left(p_1\left(t\right)+\Omega_T^{**}\right)+p_2\left(t\right)\right\} \tag{53}$$
$$\dot{p}_2^T\left(t\right) = 0 \tag{54}$$
$$\rho_t^T(\cdot) \in \max_{\rho_t(\cdot)} H\left(x^T,y^T,\rho_t(\cdot),p_1,p_2,t\right) \tag{55}$$
$$p_1^T\left(T\right)=0, \quad p_2^T\left(T\right)\geq 0 \text{ with } \; p_2^T\left(T\right)=0 \text{ if } y_T>\rho, \tag{56}$$

where (53) and (54) are adjoint equations, (55) is the maximum principle, and (56) is the transversality condition.

Proposition 3 *Given reward ρ and patent length T there exists an optimal policy $\rho_t^T(z)$ for all $t\in[0,\infty)$ with the following characteristics: i) The optimal reward function is non-stationary and increasing over time: $d\rho_t^T(z)/dt \geq 0$ for all z, ii) There exists a unique declining time dependent threshold, $\hat{z}_t^T$, such that ideas below $\hat{z}_t^T$ are non-pantentable, and ideas above $\hat{z}_t$ are patentable, and where $d\hat{z}_t^T/dt \leq 0$, iii) $\rho_t^T(z)$ converges to $\rho_T^*(z)$, the optimal reward with no active patent and $\hat{z}_t$ converges to $\hat{z}_T^*$, the optimal threshold with no*

[32] See Theorem 5 on page 107 of Seierstad an Sydsæter (1987).

active patent as t converges to T. iv) $\rho_T^*(z) < \rho^*(z)$ and $\hat{z}_T^* > \hat{z}^*$, where $\rho^*(z)$ and $\hat{z}^*$ refer to the optimal policy with free T.

An essential insight is that as $p_1^T(T) = 0$, the optimal reward $\rho_t^T(z)$ and the optimal threshold $\hat{z}_t^T$ approach the corresponding rewards and threshold in a hypothetical case with no active patent. In other words, the optimal reward function and the optimal threshold are continuous in t at T, i.e. there is no jump in policy at the point in time the patent expires. This is a standard result, any policy jump gives rise to distortions which are mitigated by "smoothing out" policy.

With no restrictions on T, we know that the optimal policy is stationary. With a fixed T, the policy is non-stationary. Obviously, to comply with the promise constraint, aggregate research over the patent period must decline, which requires that the reward be less generous, and/or the threshold be raised to at least some $t < T$. According to the proposition, over time the optimal policy path gradually converges to to the optimal stationary policy. Let us provide some intuition for this result.

Note from the maximum principle, since p_2 is fixed, it follows that $\rho_t(\cdot)$ and $\hat{z}_t$ are set to maximize

$$\begin{aligned} & u_t \mathbb{E}_{z \geq \hat{z}_t} \omega\left(\rho_t(z), z\right) + 1 - (u_t + r)\left(p_1^T(t) + \Omega_T^{**}\right) \qquad (57) \\ = \; & (u_t + r)\left[\frac{u_t \mathbb{E}_{z \geq \hat{z}_t} \omega\left(\rho_t(z), z\right) + 1}{u_t + r} - \Omega_T^{**}\right] - (u_t + r)\, p_1^T(t). \end{aligned}$$

Note that the square bracket is non-positive as Ω_T^{**} is the solution for the non-constrained optimization problem,

$$\Omega_T^{**} = \max\left[\frac{u \mathbb{E}_{z \geq \hat{z}} \omega\left(\rho(z), z\right) + 1}{u + r}\right].$$

From the control problem we know that $p_1^T(t)$ is continuous and converges to 0 as t converges to T. It is clear that as $p_1^T(t)$ converges to zero, the last term of (57) vanishes, and the optimal policy converges to the unconstrained optimal policy. As $p_1^T(t)$ is declining in t, the opportunity cost of a reward also declines in t, explaining the gradual increase in $\rho(z)$ and decline in the threshold $\hat{z}_t$ over time. Furthermore, the associated time profile of the arrival rate of patentable ideas exactly complies with the patent reward constraint, thus:

$$\int_0^T e^{-\int_0^s (u_\tau + r) d\tau} ds = \rho$$

Thus the optimal policy is non-stationary, with strong initial effective protection of the current patent, and then over time gradual softening of this protection. This yields a complexity that contrast with the optimal policy with free T, which is always stationary in optimum.

Corresponding to the free T case, if research supply is inelastic, a reward does not impact the idea arrival rate, hence the optimal reward is stationary. In this special case, the threshold is the only instrument which can be used to reduce the patentable idea arrival rate.

Finally the optimal reward function and the optimal threshold, with no active patent, differ from the corresponding optimal choices in the free T case. With one more constraint on the problem, the continuation value declines, hence $\Omega^{**} > \Omega_T^{**}$. A lower continuation value reduces the marginal benefit of a reward in a model with cumulative effects.

5 Implementability constraint

So far we have assumed that the planner observes idea quality. In this section we assume that quality is a firm's private information, observed at the point in time the idea arrives. As shown by Hopenhayn et.al. (2006), with idea quality as private information, a policy is implementable if the reward function is monotonically increasing. Since we know from Proposition 1 that the first best reward is generally non-monotone, the optimal implementable policy can be characterized by adding one more constraint to the optimization problem.

The planner solves the following maximization problem:

$$\Omega^*(\rho) = \max_{\mathcal{P}} \Omega(\rho_t(\xi)) \quad s.t.$$

$$i) \quad \int_0^T e^{-\int_0^t (\lambda_\tau \Phi_{\eta\tau} + r)d\tau} dt \leq \rho$$

$$ii) \quad \rho_t(z) \text{ is monotonically increasing in } z,$$

where ii) is the additional constraint imposed by implementability.

We can solve the control problem by replacing the control variable $\rho(z)$ in (22)-(27) with the *change* in reward as the control variable. More precisely we define a piecewise continuous control function $\delta(z) = \frac{d\rho(z)}{dz}$ and an associated starting point $\rho(0)$. As the control $\rho(z)$ may jump upwards, we also have to introduce an arbitrarily large set of k jump points, $z_1, .., z_k$. These jumps must always be positive. Both the size and location of the jumps are controlled by the planner. We refer to v_j as the size of the jump at z_j, defined as

$$v_j = \rho(z_j^+) - \rho(z_j^-), \quad j = 1, .., k,$$

where $\rho(z_j^+)$ refers to the upper limit and $\rho(z_j^-)$ the lower limit of the $\rho(z)$-function at z_j. Given this reformulation of the problem, we can solve the problem (22)-(27) with the additional monotonicity constraint.

In the general case, the choice of $\delta(\cdot)$, the jump points z_j and associated jump sizes $v_j > 0$ impact the return from research and hence the idea arrival rate, precisely as in the unrestricted model. As this dynamic effect does not add anything beyond what has already been discussed, we focus here on what the new element: describing the optimal monotone transformation of the reward function. By restricting the analysis to inelastic research, the dynamic effects are removed, and the result is derived in the simplest possible setting. It is straightforward to generalize the result to general cases with elastic research supply.

We derive the candidate policy assuming that the reward constraint does not bind. More precisely, for an arbitrary t, we derive the optimal reward profile $\rho(z)$ compatible with implementability, assuming that the idea arrival rate is compatible with a finite patent length T. If the policy is infeasible then, as before, we have one more constraint, and one more endogenous variable p_2.

Surpressing the t index yields the following control problem:

$$\begin{aligned} & \max_{\delta(\cdot)\geq 0, v^k} \int_0^{\bar{z}} \omega\left(\rho\left(\xi\right),\xi\right)\phi(\xi)d\xi && (58) \\ \text{subject to} &: \rho'(z) = \delta(z) \geq 0 && (59) \\ \rho(z_j^+) - \rho(z_j^-) &= v_j > 0 \qquad j = 0, 1, .., k && (60) \\ \rho(0) &= 0 && (61) \\ \rho_t(\bar{z}) &\leq \frac{1}{r}. && (62) \end{aligned}$$

Due to the assumptions on the θ-function, ref (2), it is trivial that the lowest possible idea $z = 0$ is non-patentable, thus we can set $\rho(0) = 0$.

We now have a standard control problem with jumps in the state variable.[33]

Let $\rho^*(z)$, $\delta^*(z)$, $z_1^*, .., z_k^*, v_1^*, .., v_k^*$ be an admissible collection which solves the problem (58) subject to (59) - (62). Furthermore, let $p_3(z)$ be a piecewise continuous adjoint function associated with (59).

The Hamiltonian is given by

$$H\left(\rho(z), \delta\left(z\right), p_3(z)\right) = \omega\left(\rho\left(z\right), z\right) + p_3\left(z\right)\delta(z) \tag{63}$$

For all non-jump points of $\rho(z)$ let $\widehat{H}$ denote the maximum value of the Hamiltonian when $\delta\left(\cdot\right)$ is chosen optimally

$$\widehat{H}\left(\rho, p_3\right) \equiv \max_{\delta(\cdot)} H\left(\rho, \delta, p_3\right). \tag{64}$$

Except for z values at which $\rho^*(z)$ is discontinuous, $p_3\left(z\right)$ is continuously differentiable and satisfies

$$p_3'\left(z\right) = -\frac{dH\left(\rho^*(z), \delta^*\left(z\right), p_3(z)\right)}{d\rho}. \tag{65}$$

The transversality condition is expressed as

$$p_3(\bar{z}) \leq 0 \ \left(= 0 \text{ if } \rho_t(\bar{z}) \ < \frac{1}{r}\right) \tag{66}$$

Furthermore at the jump points of $\rho(z)$ we have for all v_l

$$p_3(z_j^{*+}) - p_3(z_j^{*-}) = 0 \quad j = 1, .., k \tag{67}$$

and

$$p_3(z_j^{*+})\left(v_l^* - v_l\right) \geq 0 \quad l = 1, .., k \tag{68}$$

[33] See Theorem 7 on page 196 of Seierstad and Sydsæter (1987).

where $p_3(z_j^{*+})$ denotes the right-hand limit of $p_3(z)$ at z_j^*, and $p_3(z_j^{*-})$ the corresponding left-hand limit.

Finally for all v and all z at which there is no jump

$$p_3(z)v \leq 0, \quad l = 1, .., k. \tag{69}$$

From these necessary conditions can we derive two properties of the optimal reward function[34].

Proposition 4 *Given reward ρ, an optimal implementable policy $\rho_I^*(z), T_I^*$ has the following characteristics: i) in any strictly increasing segment of $\rho_I^*(z)$ we have that $d\omega(\rho_I^*(z), z)/d\rho =$* 0. *ii) any jump point $\rho_I^*(z)$ is associated with non-convexities in the $\omega(\rho, z)$ function.*

Loosely speaking, strictly increasing segments are associated with (local) maxima, and jump points are associated with discontinuities in the optimal response. Everywhere else, the optimal reward is constant.

From (67) it follows that $p_3(z)$ is continuous. Furthermore as v is non-negative, (69) implies that $p_3(z) \leq 0$. Furthermore $p_3(z_j^*) = 0$ at any jump point, since (68) must hold for all v_l. Finally from the maximum principle we have

$$\frac{dH}{d\delta(z)} = p_3(z).$$

Hence $\delta(z) = 0$ if $p_3(z) < 0$ and $\delta(z) \geq 0$ if $p_3(z) = 0$. Suppose the optimal reward function has a strictly increasing segment $[z_1, z_2]$, that is a segment in which $\delta(z) > 0$. Then $p_3(z)$ is equal to 0 in this segment, hence $p_3'(z) = 0$. Combining (63) and (65) we have

$$\frac{dH}{d\rho(z)} = \frac{d\omega}{d\rho(z)} = p_3'(z)$$

thus $\frac{d\omega(z)}{d\rho(z)} = 0$, which satisfies the condition for a (local) maxima.

Note that wherever $p_3(z) < 0$, then $\delta(z) = 0$, and the reward is fixed.

Furthermore if $\rho(z)$ jumps, then $\frac{dH}{d\rho(z_j-)} \leq 0$ and $\frac{dH}{d\rho(z_j^+)} \geq 0$. This is because we know that $p_3(z) = 0$ at a jump point, and that p_3 is always non-positive. Hence $\frac{dp_3(z_j^+)}{dz} = -\frac{dH}{d\rho(z_j^+)} = -\frac{d\omega}{d\rho(z_j^+)} \leq 0$ and $\frac{dp_3(z_j^-)}{dz} = -\frac{dH}{d\rho(z_j^-)} = -\frac{d\omega}{d\rho(z_j^-)} \geq 0$. This implies $\frac{d\omega}{d\rho(z_j^-)} \leq 0$ and $\frac{d\omega}{d\rho(z_j^+)} \geq 0$, thus any jump in the reward function is associated with non-convexities in the $\omega(\rho, z)$ as function of ρ.

The solution illustrates Myerson's concept of "ironing" the policy rule in cases where the optimal rule is non-monotonic[35]. Suppose that $\omega(\rho, z)$ is concave in ρ in each of the

[34] Note that the sufficient conditions for a solution of the problem are not satisfied. As it is trivial to show that the control problem has a solution in our case, optimal policy can be derived by comparing the continuation values associated with each candidate policy satisfying the necessary conditons. An example is given below.

[35] See e.g. Bulow and Roberts (1989) for a general discussion of "ironing out" procedures in auction theory.

segments $[0, z_b]$ and $[z_b, \bar{z}]$, which is satisfied in the iso-elastic reference case. Recall that in the benchmark case demand function $q(P)$ has constant elasticity, and the innovation size takes a Cobb Douglas form,

$$\theta(K, z) = K^{\beta} z$$

where β is a fixed positive parameter, $\beta < 1$. Imposing constant elasticities, for each non-monotone interval in the first best policy there is a fixed reward. Furthermore, as the only non-convexity is associated with the kink at z_b, there will be at most one point of discontinuity in the policy rule.[36]

Proposition 5 *In the benchmark case the optimal implementable policy is a two point policy $\rho_M \geq \rho_L$, where ρ_M is the reward of major ideas, and ρ_L is the reward for minor ideas.*

Let us now provide intution to this result. With inelastic research the optimal reward maximizes the continuation value (16) for each possible idea quality z,

$$\omega\left(\rho\left(z\right), z\right) = \theta\left(\rho\left(z\right), z\right)\left[\Omega^{*}\left(\rho\left(z\right)\right) - \rho\left(z\right)\left(1 - c\right)\right],$$

recalling that $\Omega^{*}\left(\rho\left(z\right)\right)$ is the gross continuation value, $\rho\left(z\right)\left(1 - c\right)$ is the Posner deadweigth loss, and $\theta\left(\rho\left(z\right), z\right)$ is the innovation size.

A reward impacts continuation value $\omega\left(\rho\left(z\right), z\right)$ in four different ways. First a higher $\rho(z)$ stimulates development investments, thus $\theta\left(\rho\left(z\right), z\right)$ increases, and due to the cumulative effect, proportionally scales up the continuation value – this is the incentive power effect discussed in the previous section. Second, if the reward is set so high that it binds, then $\Omega^{*}\left(\rho\left(z\right)\right)$ is declining in $\rho\left(z\right)$, as it gives rise to future distortions. This is captured by the adjoint variable associated with the control problem,

$$\frac{d\Omega^{*}\left(\rho\left(z\right)\right)}{d\rho(z)} = p_2(\rho(z)) < 0.$$

As $p_2(\rho(z))$ is declining in $\rho(z)$, a higher reward makes distortions more severe.

Third, an increase in $\rho(z)$ increases the cost associated with the Posner deadweight loss since the monopoly duration increases, hence consumers suffer more. This is the standard welfare loss of monopoly.

The fourth and final effect is associated with *rent extraction.* A higher reward increases $\theta\left(\rho\left(z\right), z\right)$ and thus allows the patentee to raise its price, and thus transfer rent from consumers to the firm. Due to full rent dissipation on the firm's side, this transfer is not neutral but has a negative effect on welfare. Note that this fourth effect is unique for limit pricing.

The optimal reward of major innovations tends to be inelastic in idea quality, which can be explained as follows. The incentive power effect indicates a monotonically increasing reward function. The complementarity between idea quality z and development investments K

[36]In the general case, there might be local maxima and non-convexities in the ω function which implies that in non-monotone intervals in the first best reward function the "ironing out" procedure does not yield a fixed reward, but a monotonically increasing reward.

implies that the gross return on investment increases as quality increases. Providing stronger protection for better ideas stimulates overall development investment. This effect explains the monotonicity of the optimal reward function in Hopenhayn et.al. (2006). However the deadweight loss associated with patent policy, as well as the losses associated with future distortions, are higher for better ideas. With the constant elasticity specification, $\theta(K,z) = K^{\beta}z$, the positive and negative effects cancel each other out, and the optimal reward of major ideas is fixed, independent of idea quality z.

For minor innovations the same intuition applies, except for one particular effect, the magnitude of the Posner deadweight loss. As the monopoly price exceeds the limit price, the cost of extending the monopoly period shrinks as the limit price declines, and eventually approaches zero as the limit price approaches marginal cost.

In addition, as the set of patentable ideas increases, the zero profit condition is eventually binding, i.e. $\rho = \rho^{\min}(z)$ for small values of z, ref. figure 1. Including more ideas requires then a higher reward; $\rho^{\min}(z)$ increases as z declines.

Furthermore, in the constant elasticity case, $\omega\left(\rho(z), z\right)$ is concave in $\rho(z)$ in each segment $\rho(z) \in \left[\rho^{\min}(z), \rho_b(z)\right]$ and $\rho(z) \in \left[\rho_b(z), \frac{1}{r}\right]$. Thus it follows from Proposition 3 that the optimal reward function has at most one jump point, which is the z level that distinguishes major from minor innovations. Finally, since the first best reward is constant for major innovations and declining for major innovations, it is clear from Proposition 3 that the optimal implementable policy is a two point policy.

6 Concluding remark

This paper derives the optimal patent policy in an environment with sequential innovations. It is shown that the optimal policy is stationary if and only if patent length is endogenously determined. If patent length is exogenously determined, the optimal policy is non-stationary.

The optimal policy is non-monotone. This creates an implementability problem if innovation quality is private information. Under standard assumptions about innovation and demand structure, the optimal implementable policy is shown to be a two point policy.

7 Appendix

Derivation of Bellman equation

Let $V_n\left(\rho_t\left(z\right), z\right)$ denote the continuation value for the planner immediately *after* a new patent z is approved with promised patent duration $\rho_t\left(z\right)$ *and* the development investment $K_n\left(\rho_t\left(z\right), z\right)$ is made. Then we can write the planner's optimization problem as

$$V_n\left(\rho_n, z_n\right) = \max_{\mathcal{P}_n} \left\{ \begin{array}{c} \int_0^{\infty} \left\{u_t \mathbb{E}_{z \in \Upsilon_t}\left[V_{n+1}\left(\rho_t\left(\xi\right), \xi\right) - K_{n+1}\left(\rho_t\left(\xi\right), \xi\right)\right]\right\} e^{-\int_0^t (u_\tau + r) d\tau} dt \\ + \int_0^T \left[c_n\left(\rho_n, z_n\right) + \pi_n\left(\rho_n, z_n\right) - b\left(\lambda_t(\cdot)\right)\lambda_t(\cdot)\right] e^{-\int_0^t (u_\tau + r) d\tau} dt \\ + \int_T^{\infty} \left[s_n\left(\rho_n, z_n\right) - b\left(\lambda_t(\cdot)\right)\lambda_t(\cdot)\right] e^{-\int_0^t (u_\tau + r) d\tau} dt \end{array} \right\}. \tag{70}$$

Condition (13) implies

$$\int_0^\infty u_t \mathbb{E}_{z\in\Upsilon_t}\left[\rho_t\left(\xi\right)\pi_{n+1}\left(\rho_t(\xi),\xi\right)-K_{n+1}\left(\rho_t(\xi),\xi\right)\right]e^{-\int_0^t(u_\tau+r)d\tau}dt=\int_0^\infty b\left(\lambda_t(\cdot)\right)\lambda_t(\cdot)e^{-\int_0^t(u_\tau+r)d\tau}dt$$

Inserted in (70), and define $W_n\left(\rho_t\left(z\right),z\right)\equiv V_n\left(\rho_t\left(z\right),z\right)-\rho_t\left(z\right)\pi_n\left(\rho_t(z),z\right)$ yields the expression in the main text.

Proof Proposition 1

i) Let us first verify that a stationary policy with $\rho_t^*\left(\cdot\right)=\rho^*\left(\cdot\right)$ is optimal. To see this, note that (31) implies that $p_2\left(t\right)=p_2$, where p_2 is some constant. Let $p_1\left(t\right)=p_1$ where p_1 is a constant. The adjoint equation (30) becomes

$$\lambda\left(\rho^*\left(\cdot\right)\right)\int_{\xi\in\Upsilon_t}\left[\omega\left(\rho^*\left(\xi\right),\xi\right)-p_1\right]\phi\left(\xi\right)d\xi+1-p_1r+p_2=0.$$

The maximum principle (32) satisfies (note that $u\equiv\lambda\left(\rho\left(\cdot\right)\right)\left(1-\Phi(\hat{z})\right)$

$$\rho^*(\cdot)\text{ maximizes }\lambda\left(\rho\left(\cdot\right)\right)\int_{z\geq\hat{z}}\left[\omega\left(\rho\left(\xi\right),\xi\right)-p_1\right]\phi\left(\xi\right)d\xi. \tag{71}$$

Since the expression within the max operator depends on t only through $\rho_t\left(\cdot\right)$, a stationary policy is optimal.

ii) It follows from the maximum principle that $\rho(z)\geq\rho^{\min}(z)$, thus $\eta(z)=1$, if there exists an $\rho(z)\geq\rho^{\min}(z)$ such that

$$\max_{\rho(z)\geq\rho^{\min}(z)}H\left(x^*,y^*,\rho^*_{\xi\neq z}(\cdot),\rho(z),p_1^*,p_2^*\right)\geq H\left(x^*,y^*,\rho^*_{\xi\neq z}(\cdot),0,p_1^*,p_2^*\right) \tag{72}$$

Removing common terms (72) can be written

$$\lambda\left(\rho^*_{\xi\neq z}(\cdot),\rho(z)\right)\left[\int_{z\in\Upsilon}\left[\omega\left(\rho\left(\xi\right),\xi\right)-p_1\right]\phi(\xi)d\xi\right]\geq\lambda\left(\rho^*_{\xi\neq z}(\cdot),0\right)\left[\int_{z\in\Upsilon,\eta(z)=0}\left[\omega\left(\rho_t\left(\xi\right),\xi\right)-p_1\right]\phi(\xi)d\xi\right]$$

we have

$$\lambda\left(\rho^*_{\xi\neq z}(\cdot),\rho(z)\right)=\lambda\left(\rho^*_{\xi\neq z}(\cdot),0\right)+\int_{\rho^{\min}(z)}^{\rho(z)}\frac{\partial\lambda\left(\rho^*_{\xi\neq z}(\cdot),\rho\right)}{\partial\rho}d\rho$$

inserted yields the following condition for $\eta_t(z)=1$:

$$\left[\int_{z\in\Upsilon}\left[\omega\left(\rho^*\left(\xi\right),\xi\right)-p_1\right]\phi(\xi)d\xi\right]\int_{\rho^{\min}(z)}^{\rho(z)}\frac{\partial\lambda\left(\rho^*_{\xi\neq z}(\cdot),\rho\right)}{\partial\rho}d\rho$$
$$+\lambda\left(\rho^*_{\xi\neq z}(\cdot),0\right)\left[\omega\left(\rho^*\left(z\right),z\right)-p_1\right]\phi(z)\geq 0$$

Differentiating the free entry condition for research yields

$$\frac{d\lambda}{d\rho(z)}=\frac{1}{b'(\lambda)}\theta\left(\rho\left(z\right),z\right)\phi(z)\pi(z)>0$$

thus the planner rewards all ideas for which

$$\omega\left(\rho^{*}\left(z\right),z\right)-p_{1}+\left[\int_{z\in\Upsilon}\left[\omega\left(\rho^{*}\left(\xi\right),\xi\right)-p_{1}\right]\phi(\xi)d\xi\right]\frac{\int_{\rho^{\min}(z)}^{\rho^{*}(z)}\pi(z)\theta\left(\rho,z\right)d\rho}{b'(\lambda)\lambda}\geq 0 \qquad (73)$$

Note that the partial derivatives $\partial\omega\left(\rho^{*}\left(z\right),z\right)/\partial z$ and $\partial\left[\theta\left(\rho,z\right)\pi(z)\right]/\partial z$ are strictly positive. Suppose the optimal policy is not a treshold-policy. Denote by $\tilde{z}$ the smallest patentable idea, and let $\mathbf{z}$ be the set of non-patentable ideas above $\tilde{z}$. Consider the following deviation policy: denote by $\tilde{z}'$ a new treshold, $\tilde{z}'>\tilde{z}$. Provide all ideas in $\mathbf{z}$ with patents replicating the reward levels provided to ideas in the segment $[\tilde{z},\tilde{z}']$, and adjust $\tilde{z}'$ such that the arrival rate of patentable ideas is unchanged. Since $\omega\left(\rho^{*}\left(z\right),z\right)$ and $\pi(z)\theta\left(\rho,z\right)$ are increasing in z given $\rho^{*}\left(z\right)$, the deviation improves consumer welfare.

iii) Discussed in the main text, section 3.3.

iv) Discussed in the main text, sections 3.1 and 3.2.

Equation (45)

Differentiating the free entry condition yields

$$\Delta\lambda(\hat{z})=\frac{\int_{\rho^{\min}(\hat{z})}^{\rho(\hat{z})}\theta\left(\rho,\hat{z}\right)d\rho}{EL_{\lambda}b(\lambda\left(\rho^{*}\left(\cdot\right)\right)}\frac{\pi(\hat{z})}{b(\lambda\left(\rho^{*}\left(\cdot\right)\right)}>0$$

Proof Lemma 1

Note that if T_n is finite then $p_2=0$, thus from (44) and (38) we have

$$\omega\left(\rho\left(\hat{z}\right),\hat{z}\right)\leq p_{1}=\Omega^{**}$$

with strong inequality if the research supply is elastic. Suppose to the contrary that $T_{n+1}(\hat{z})$ is finite. Then

$$\omega\left(\rho\left(\hat{z}\right),\hat{z}\right)=\theta\left(\rho\left(\hat{z}\right),\hat{z}\right)\left[\Omega^{**}-\rho\left(1-c\right)\right]$$

We can show that

$$\theta\left(\rho\left(\hat{z}\right),\hat{z}\right)\left[\Omega^{**}-\rho\left(1-c\right)\right]>\Omega^{**}$$

which yields a contradiction. To see this, note that 1) $c\geq 1/\theta\left(\rho\left(\hat{z}\right),\hat{z}\right)$, as the consumer can guarantee itself the social surplus associated with the old product, and 2) $\rho\leq 1/r$. Hence

$$\begin{aligned}&\theta\left(\rho\left(\hat{z}\right),\hat{z}\right)\left[\Omega^{**}-\rho\left(1-c\right)\right]\\ >\ &\theta\left(\rho\left(\hat{z}\right),\hat{z}\right)\Omega^{**}-\theta\left(\rho\left(\hat{z}\right),\hat{z}\right)\rho+\rho>\Omega^{**}\end{aligned}$$

thus

$$\theta\left(\rho\left(\hat{z}\right),\hat{z}\right)\left[\Omega^{**}-\rho\left(1-c\right)\right]>\Omega^{**}$$

Since

$$\theta\left(\rho\left(\hat{z}\right),\hat{z}\right)\left[\Omega^{**}-\rho\left(1-c\right)\right]>\theta\left(\rho\left(\hat{z}\right),\hat{z}\right)\left[\Omega^{**}-\rho\right]+\rho>\Omega^{**}$$

which yields a contradiction

Proof of Proposition 2

The optimal development investment is

$$K(\rho, z) = \arg\max_{K} \rho\theta(K, z)\pi(P) - K$$

$$s.t. \ \ \theta(K, z) \int_{P}^{\infty} q(\zeta)\, d\zeta \geq 1$$

If the constraint binds, the first order condition is:

$$\rho \left[\theta_K(K, z)\pi(P) + \theta(K, z)\pi'(P) \frac{dP}{dK} \right] = 1$$

Which can be written:

$$\rho\theta_K(K, z) \left[\pi(P) + \frac{\theta(K, z)}{\theta_K(K, z)} \pi'(P) \frac{dP}{dK} \right] = 1 \tag{74}$$

From the reward constraint $\theta(K, z) \int_P^\infty q(\zeta)\, d\zeta = 1$ it follows that

$$\frac{dP}{dK} = \frac{\theta_K(K, z)}{\theta(K, z)} \frac{c(P)}{q(P)}$$

Furthermore

$$\pi'(P) = q(P) \left[1 - \left(1 - \frac{mc}{P} \right) \varepsilon(P) \right]$$

hence (74) can be written

$$\rho\theta_K(K, z) \left[\pi(P) + \left[1 - \left(\frac{P - mc}{P} \right) \varepsilon(P) \right] c(P) \right] = 1$$

Since P is increasing in z and ρ, it is sufficient to show that $\pi(P) + \left[1 - \left(\frac{P-mc}{P} \right) \varepsilon(P) \right] c(P)$ is declining in P. Differenting I yields, where we note that $c'(P) = -q(P)$ and $\pi(P) = q(P) + (P - mc)q'(P)$

$$sgn \frac{dI}{dP} = -\frac{mc}{P^2} \varepsilon(P) - \left(1 - \frac{mc}{P} \right) \varepsilon'(P).$$

Thus the condition for $dI/dP < 0$ corresponds to the standard sufficient conditions for the unconstrained monopoly price problem.

p_1 is increasing in ρ

Suppose not. Then from Proposition 1 it follows that $\rho^*(z)$ is increasing in ρ and $\hat{z}$ is declining. Hence the arrival rate of patentable ideas increases, incompatible with a higher binding reward constraint.

The second term of (44)

Differentiating the free entry condition (13) yields

$$\frac{\partial \lambda}{\partial \rho(z)} = \frac{\lambda\theta(\rho(z), z)}{EL_\lambda b(\lambda(\rho^*(\cdot))} \frac{\pi(P)}{b(\lambda(\rho^*(\cdot))} \phi(z)$$

Calculations based on iso elastic demand and innovation step

Let the market demand function be specified as

$$q(P) = P^{-\varepsilon}$$

and we scale marginal cost such that maximum social surplus equals one:

$$mc = \left(\frac{1}{\varepsilon - 1}\right)^{\frac{1}{\varepsilon-1}}$$

The monopoly price is the standard constant markup pricing rule

$$P^M = \frac{\varepsilon}{\varepsilon - 1} mc$$

which yields equilibrium profit

$$\pi_0 = \left(\frac{\varepsilon - 1}{\varepsilon}\right)^{\varepsilon}$$

and consumer surplus

$$c_0 = \int_{P^M}^{\infty} \zeta^{-\varepsilon} d\zeta = \left(\frac{\varepsilon - 1}{\varepsilon}\right)^{\varepsilon - 1}$$

Development investments: if $\rho \geq \rho_b(z)$ the incumbent's maximizes

$$\begin{aligned} & \rho\theta(K, z)\pi_M - K \\ = \; & \rho K^{\beta} z \pi_M - K \end{aligned}$$

with first order condition

$$\rho\beta K^{\beta-1} z \pi_M - 1 = 0$$

which determines $K^M(\rho, z)$, and from $\theta(\rho, z) := \theta(K(\rho, z), z)$ we obtain

$$\theta_M(\rho, z) = \left(\beta\left(\frac{\varepsilon - 1}{\varepsilon}\right)^{\varepsilon}\right)^{\frac{\beta}{1-\beta}} \rho^{\frac{\beta}{1-\beta}} z^{\frac{1}{1-\beta}}$$

If $\rho < \rho_b(z)$, the consumer is indifferent between buying the new product at price P and the outside option, thus

$$c(P) = \frac{u(z)}{\theta}$$

which links the limit price to innovation size θ

$$P(\theta, u(z)) = \left(\frac{1}{\varepsilon - 1} \frac{\theta}{u(z)}\right)^{\frac{1}{\varepsilon-1}}$$

The firm's profit is

$$\pi_L = (\varepsilon - 1)\left(\left(\frac{\theta}{u(z)}\right)^{\frac{1}{\varepsilon-1}} - 1\right)\left(\frac{\theta}{u(z)}\right)^{-\frac{\varepsilon}{\varepsilon-1}}$$

The firm maximizes

$$\begin{aligned} & \rho\theta\pi_L - K \\ = \quad & \rho\left(\varepsilon-1\right)\left(u(z) - \frac{\theta^{-\frac{1}{\varepsilon-1}}}{u(z)^{-\frac{\varepsilon}{\varepsilon-1}}}\right) - K \end{aligned}$$

with first order condition

$$\rho\left(\frac{u(z)}{\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}}\frac{d\theta}{dK} - 1 = 0$$

Inserting from $\theta = K^{\beta}z$ yields

$$K^L = \left(\frac{\rho\beta u(z)^{\varepsilon}}{z}\right)^{\frac{1}{\varepsilon-1+\beta}}$$

hence innovation step

$$\theta_L = (\rho\beta)^{\frac{(\varepsilon-1)\beta}{\varepsilon-1+\beta}} z^{\frac{\varepsilon-1}{\varepsilon-1+\beta}} u(z)^{\frac{\varepsilon\beta}{\varepsilon-1+\beta}}$$

The idea level z_D that delineates drastic from non-drastic innovations, $\theta_M = \theta_L$, which yields

$$z_D(\rho) = \frac{\left(\frac{\varepsilon}{\varepsilon-1}\right)^{\varepsilon-1+\beta}}{\beta^{\beta}\rho^{\beta}} u(z)^{1-\beta}$$

8 References

Acemoglu, Daron, and Ufuk Akcigit. "Intellectual property rights policy, competition and innovation." Journal of the European Economic Association 10.1 (2012): 1-42.

Acemoglu, Daron, and Dan Cao. "Innovation by entrants and incumbents." Journal of Economic Theory 157 (2015): 255-294.

Aghion, Philippe, and Peter Howitt. “A Model of Growth Through Creative Destruction”. Econometrica 60.2 (1992): 323–351

Aghion, Philippe, Peter Howitt, and Susanne Prantl. "Patent rights, product market reforms, and innovation." Journal of Economic Growth 20.3 (2015): 223-262.

Banal-Estañol, Albert, and Inés Macho-Stadler. "Scientific and commercial incentives in R&D: research versus development?." Journal of Economics & Management Strategy 19.1 (2010): 185-221.

Bessen, James, and Eric Maskin. "Sequential innovation, patents, and imitation." The RAND Journal of Economics 40.4 (2009): 611-635.

Bulow, Jeremy, and John Roberts. "The simple economics of optimal auctions." The Journal of Political Economy (1989): 1060-1090.

Chang, Howard F. "Patent scope, antitrust policy, and cumulative innovation." The RAND Journal of Economics (1995): 34-57.

Cornelli, Francesca, and Mark Schankerman. "Patent renewals and R&D incentives." The RAND Journal of Economics (1999): 197-213.

Denicolò, Vincenzo. "Ecomonic Theories of the Nonobviousness Requirement for Patentability: A Survey." Lewis & Clark L. Rev. 12 (2008): 443.

Erkal, Nisvan, and Suzanne Scotchmer. "Scarcity of Ideas and R&D Options: Use it, Lose it or Bank it." No. w14940. National Bureau of Economic Research, 2009.

Gallini, Nancy T. "Patent policy and costly imitation." The RAND Journal of Economics (1992): 52-63.

Gallini, Nancy, and Suzanne Scotchmer. "Intellectual property: when is it the best incentive system?." Innovation Policy and the Economy, Volume 2. MIT Press, 2002. 51-78.

Gilbert, Richard, and Carl Shapiro. "Optimal patent length and breadth." The RAND Journal of Economics (1990): 106-112.

Hopenhayn, Hugo, Gerard Llobet, and Matthew Mitchell. "Rewarding sequential innovators: Prizes, patents, and buyouts." Journal of Political Economy 114.6 (2006): 1041-1068.

Hopenhayn, Hugo, and Matthew Mitchell. "Optimal Patent Policy with Recurrent Innovators." UCLA and University of Toronto Working paper (2011).

Hunt, Robert M. "Patentability, industry structure, and innovation." The Journal of Industrial Economics 52.3 (2004): 401-425.

Klemperer, Paul. "How broad should the scope of patent protection be?." The RAND Journal of Economics (1990): 113-130.

Klette, Tor Jakob, and Samuel Kortum. "Innovating Firms and Aggregate Innovation." Journal of Political Economy 112.5 (2004).

Kremer, Michael. "Patent Buyouts: A Mechanism for Encouraging Innovation." The Quarterly Journal of Economics 113.4 (1998): 1137-1167.

O'Donoghue, Ted. "A patentability requirement for sequential innovation." The RAND Journal of Economics (1998): 654-679.

O'Donoghue, Ted, Suzanne Scotchmer, and Jacques-François Thisse. "Patent breadth, patent life, and the pace of technological progress." Journal of Economics & Management Strategy 7.1 (1998): 1-32.

Hunt, Robert M. "Patentability, industry structure, and innovation." The Journal of Industrial Economics 52.3 (2004): 401-425.

Merges, Robert P., and Richard R. Nelson. "On the complex economics of patent scope." Columbia Law Review 90.4 (1990): 839-916.

Nordhaus,W. D., Invention, Growth, and Welfare; A Theoretical Treatment of Technological Change, Cambridge, Mass. (1969), ch. 5.

Parello, Carmelo, and Luca Spinesi. "Partial imitation, inequality and growth: the role of the courts' interpretation of patent law." Innovation, Unemployment, and Policy in the Theories of Growth and Distribution (2005): 110.

Posner, Richard A. "The Social Costs of Monopoly and Regulation." The Journal of Political Economy, Vol. 83, No. 4 (1975), 807-828

Sannikov, Yuliy. "A continuous-time version of the principal-agent problem." The Review of Economic Studies 75.3 (2008): 957-984.

Scherer, Frederic M. "Nordhaus' theory of optimal patent life: A geometric reinterpretation." The American Economic Review 62.3 (1972): 422-427.

Scotchmer, Suzanne. "Standing on the shoulders of giants: cumulative research and the patent law." The Journal of Economic Perspectives 5.1 (1991): 29-41.

Scotchmer, Suzanne. "Protecting early innovators: should second-generation products be patentable?." The Rand Journal of Economics (1996): 322-331.

Scotchmer, Suzanne. "Ideas and Innovations: Which should be subsidized?." Available at SSRN 1755091 (2011).

Scotchmer, Suzanne, and Jerry Green. "Novelty and disclosure in patent law." The RAND Journal of Economics (1990): 131-146.

Seierstad, Atle, and Knut Sydsaeter. Optimal control theory with economic applications. Elsevier North-Holland, Inc., 1986.

Sena, Vania. "The return of the Prince of Denmark: A survey on recent developments in the economics of innovation." The Economic Journal 114.496 (2004): F312-F332.

Sorek, Gilad. "Patents and quality growth in OLG economy." Journal of Macroeconomics 33.4 (2011): 690-699.

Spear, Stephen E., and Sanjay Srivastava. "On repeated moral hazard with discounting." The Review of Economic Studies 54.4 (1987): 599-617.

Tandon, Pankaj. "Optimal patents with compulsory licensing." The Journal of Political Economy (1982): 470-486.